

Universal Approximation Theorems for Differentiable Geometric Deep Learning

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Abstract

This paper addresses the growing need to process non-Euclidean data, by introducing a geometric deep learning (GDL) framework for building universal feedforward-type models compatible with differentiable manifold geometries. We show that our GDL models can approximate any continuous target function uniformly on compact sets of a controlled maximum diameter. We obtain curvature dependant lower-bounds on this maximum diameter and upper-bounds on the depth of our approximating GDL models. Conversely, we find that there is always a continuous function between any two non-degenerate compact manifolds that any “locally-defined” GDL model cannot uniformly approximate. Our last main result identifies data-dependent conditions guaranteeing that the GDL model implementing our approximation breaks “the curse of dimensionality.” We find that any “real-world” (i.e. finite) dataset always satisfies our condition and, conversely, any dataset satisfies our requirement if the target function is smooth. As applications, we confirm the universal approximation capabilities of the following GDL models: [Ganea et al. \(2018\)](#)’s hyperbolic feedforward networks, the architecture implementing [Krishnan et al. \(2015\)](#)’s deep Kalman-Filter, and deep softmax classifiers. We build universal extensions/variants of: the SPD-matrix regressor of [Meyer et al. \(2011b\)](#), and [Fletcher et al. \(2009\)](#)’s Procrustean regressor. In the Euclidean setting, our results imply a quantitative version of [Kidger and Lyons \(2020\)](#)’s approximation theorem and a data-dependent version of [Yarotsky and Zhevnerchuk \(2020\)](#)’s uncursed approximation rates.

Keywords: Geometric Deep Learning, Symmetric Positive-Definite Matrices, Hyperbolic Neural Networks, Deep Kalman Filter, Shape Space, Riemannian Manifolds, Curse of Dimensionality.

1. Introduction

Since their introduction in [McCulloch and Pitts \(1943\)](#), the approximation capabilities of neural networks and their superior efficiency over many classical modelling approaches have led them to permeate many applied sciences, many areas of engineering, computer science, and applied mathematics. Nevertheless, the complex geometric relationships and interactions between data from many of these areas are best handled with machine learning models designed for processing and predicting from/to such non-Euclidean structures.

This need is reflected by the emerging machine learning area known as *geometric deep learning*. A brief (non-exhaustive) list of situations where geometric (deep) learning is a key tool includes: shape analysis in neuroimaging (e.g. [Thomas Fletcher P. \(2013\)](#) and [Masci et al. \(2020\)](#)), human motion patterns learning (e.g. [Yan et al. \(2018\)](#), [Jain et al. \(2016\)](#), and [Huang et al. \(2017\)](#)), molecular

fingerprint learning in biochemistry (e.g. [Duvenaud et al. \(2015\)](#)), predicting covariance matrices (e.g. [Meyer et al. \(2011a\)](#)), robust matrix factorization (e.g. [Baes et al. \(2019\)](#) and [Herrera et al. \(2020\)](#)), learning directions of motion for robotics using spherical data as in (e.g. [Dai and Müller \(2018b\)](#), [Straub et al. \(2015\)](#), and [Dutordoir et al. \(2020\)](#)), and many other learning problems.

There are many structures of interest in geometric deep learning such as differentiable manifolds (overviewed in [Bronstein et al. \(2017\)](#) and in [Bronstein et al. \(2021\)](#) as well as in our applications Sections 3.4 and 4.4), graphs ([Zhou et al. \(2020\)](#)), deep neural networks with group invariances or equivariances (respectively, see [Yarotsky \(2021\)](#) and [Cohen and Welling \(2016\)](#)). This paper concentrates on the first of these cases and develops a general theory compatible with any differentiable manifold input and output space. To differentiate the manifold-valued setting from the other geometric deep learning problems just described, we fittingly refer to it as “*differentiable geometric deep learning*”.

Contributions

This paper adds to this rapidly growing research area by developing a self-contained geometric deep learning framework for building universal deep neural models between differentiable manifolds. The models in our framework are all universal and are constructed explicitly from simpler building blocks. Additionally, each of our universal (resp. efficient) approximation theorems is quantitative.

After introducing our framework and presenting our main results, we demonstrate the scope and flexibility of our proposed framework by using it to validate the approximation capabilities of various commonly used geometric deep learning models. These include: *the Hyperbolic Feedforward Networks* of [Ganea et al. \(2018\)](#) for learning from efficient embeddings of large undirected graphs (see [Munzner \(1997\)](#)), trees (see [Sala et al. \(2018\)](#)), complex social-networks (see [Krioukov et al. \(2010\)](#)), and hierarchical datasets (see [Nickel and Kiela \(2017\)](#)), the architecture implemented in the *Deep Kalman filter* of [Krishnan et al. \(2015\)](#) for approximating update rules between spaces of non-degenerate Gaussian measures, and deep softmax classifiers which are omnipresent in contemporary multiclass classification. We then show how our framework yields universal extensions to the following popular non-Euclidean regression models: the Procrustean pre-shape space regression models of [Thomas Fletcher P. \(2013\)](#) used in biomedical imaging, and computer-vision and its analogue in the projective shape space of [Mardia and Patrangenaru \(2005\)](#), and the symmetric positive-definite matrix regressor of [Meyer et al. \(2011b\)](#) (see [Pennec et al. \(2006\)](#) for uses of this geometry in tensor computing). Illustrations of our results in the context of spherical and toral input and output spaces are also provided; we note that spherical data is prevalent in astronomy applications (see [Fisher et al. \(1993\)](#)), and toral geometries have found recent applications in data visualization (e.g. [Maron et al. \(2017\)](#) and [Li \(2004\)](#)).

In the Euclidean context, our results imply a *quantitative version* of the qualitative universal approximation theorem for deep and narrow feedforward networks (recently obtained in [Kidger and Lyons \(2020\)](#)) as well as an extension of the dimension-free approximation rates of [Yarotsky and Zhevnerchuk \(2020\)](#) for non-smooth functions defined on “efficient datasets” (introduced in Section 3.3.2). We also find that all datasets are efficient when the target function is smooth and, conversely, any “real-world” dataset (i.e., non-empty and finite) is efficient for every target function. In particular, this last result offers a new “datacentric” perspective explaining the well-documented effectiveness of deep learning.

Our Approach

Often, geometric machine learning models work by first linearizing the non-Euclidean data in the input space via a *feature map* $\phi : \mathcal{X} \rightarrow \mathbb{R}^p$, then processing the linearized data via a classical “Euclidean” learning model g , and finally using an inverted linearization step to recover non-Euclidean predictions in the output space \mathcal{Y} via some “*readout map*” $\rho : \mathbb{R}^m \rightarrow \mathcal{Y}$. This schema, illustrated

$$\begin{array}{ccc}
 \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\
 \downarrow \phi & & \uparrow \rho \\
 \mathbb{R}^p & \xrightarrow{g} & \mathbb{R}^m
 \end{array}$$

Figure 1: Lifting Euclidean learning models to non-Euclidean input/output spaces.

in Figure 1, has been successfully employed (either explicitly or implicitly) in various areas of machine learning; examples include the principal geodesic analysis method of Fletcher (2013), the Log-Euclidean Kernel regressors of Li et al. (2013), the unscented Kalman filters of Hauberg Søren et al. (2013), the deep Kalman filter of Krishnan et al. (2015), the hyperbolic neural network models of Ganea et al. (2018) and of Shimizu et al. (2021), the architecture implemented in the feature-map learning meta-algorithm of Kratsios and Hyndman (2021), and others.

More generally, in Kratsios and Bilokopytov (2020), it was shown that when g in Figure 1 is allowed to be any universal learning model class from \mathbb{R}^p to \mathbb{R}^m then, under certain conditions on $\phi : \mathcal{X} \rightarrow \mathbb{R}^p$ and on $\rho : \mathbb{R}^m \rightarrow \mathcal{Y}$, any continuous function $f : \mathcal{X} \rightarrow \mathcal{Y}$ could be approximated uniformly on compact subsets of \mathcal{X} . However, these conditions on $\rho : \mathbb{R}^m \rightarrow \mathcal{Y}$ require the “global geometry of \mathcal{Y} ” to be “approximately Euclidean”. Unfortunately, this leaves many interesting input/output spaces arising naturally in geometric deep learning applications out of the scope of this type of approach.

We take this observation as our starting point. This paper offers a complete solution to the problem of (explicitly) developing universal deep neural models between arbitrary differentiable manifolds \mathcal{X} and \mathcal{Y} .

Our Differentiable Geometric Deep-Learning Framework

Our analysis begins with the observation that universal approximation between differentiable manifolds is *necessarily a local problem* (in Theorems 5 and 6). Thus, we reinterpret Figure 1 as only holding “locally”, instead of “globally”. This is achieved by letting the maps ϕ and ρ vary and be fully-specified by the local geometries of \mathcal{X} and \mathcal{Y} , respectively. Our main geometric deep learning model, the *geometric deep networks* (GDNs), are succinctly summarized by Figure 2.

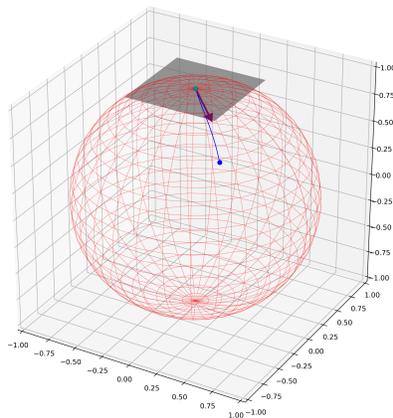


Figure 2: Visualization representation of GDNs

Our GDN models are build from three distinct types of layers. Fix a **feedforward network** g defined in the **tangent space** of a fixed reference point x on \mathcal{X} . First, the local non-Euclidean data near x is fed into this “tangential” **feedforward network** g by a local-linearization procedure which sends points near x to the **velocity vector** of an optimally efficient curve emanating from x and terminating at the data-point. Next, this velocity vector is processed by the **feedforward network** g . Lastly, the output of **the network** g is mapped onto the output space, about some reference point $y \in \mathcal{Y}$, by an analogous but inverted linearization layer.

Remark 1 *Our description of the GDN model, summarized by Figure 2, requires fixing a complete Riemannian metric. These tools will be overviewed shortly.*

The GDN framework is flexible and general enough to approximate any function between differentiable manifolds. Nevertheless, at times, it may be more convenient/natural to use GDNs as an integral component of geometric deep learning models with more sophisticated computational graphs. In the second part of this paper, we build a general class of such models whose additional layers implement fundamental geometric operations such as quotients (useful for easily encoding symmetries and invariances), products (useful for parallelizing GDNs defined on potentially different input and output spaces), and feature as well as readout maps which can be used to combine these models as well as process more general “non-smooth/singular” geometries. Examples of the latter are manifolds with boundaries/corners, e.g. the standard simplex.

Organization of Paper

This paper is organized as follows. Section 2 overviews the relevant topological and Riemannian geometric background required for the presentation of our main results. Section 3 contains our main results surrounding the approximation capabilities of GDNs. These begin with a necessary condition for universal approximation between complete Riemannian manifolds and the impossibility of models defined from local data to approximate arbitrary continuous functions between compact Riemannian manifolds of positive dimension uniformly on *arbitrary compact subsets of the input space*. This motivates us to look for a “controlled universal approximation theorem” which focuses on approximation of continuous functions between Riemannian manifold uniformly on compact subsets of a given maximum diameter.

Next, we confirm that this is an appropriate notion of universal approximation for geometric deep learning by deriving our “controlled universal approximation theorem” for GDNs, which is *quantitative* on two fronts: it provides a (non-trivial) curvature-dependant lower-bound on the maximum diameter of a compact subset of \mathcal{X} on which a function of a given regularity can be uniformly approximated, and second, it provides detailed depth-order estimates of the deep and narrow GDN implementing the approximation. Next, we obtain dimension-free approximation rates by GDNs on efficient datasets (introduced in Section 3.3.2). We show that any “real-world dataset” (i.e., non-empty and finite subset of \mathcal{X}) is efficient for any function (not necessarily continuous) and, conversely, we find that any dataset is efficient for every smooth function with Lipschitz higher-order partial derivatives. In Section 3.4, quantitative universal approximation theorems for many of the above geometric deep learning architectures are derived.

Section 4 develops the approximation capabilities, and the calculus for the geometric deep neural models build from the GDNs using the “geometric processing layers” described above. Section 4.4 derives the universal approximation theorems for the remaining geometric deep learning architectures described in the introduction (as well as some others for illustrative purposes).

Notation and Standing Assumptions

We denote by $\mathcal{NN}_{p,m,k}^\sigma$ the class of functions described by feedforward neural networks with p neurons in the input layer, m neurons in the output layer, and an arbitrary number of hidden layers,

each with at-most k neurons and with activation function σ . Thus, $f \in \mathcal{NN}_{p,m;k}^\sigma$ if there is a $J \in \mathbb{N}_+$ and there are composable affine maps W_{J+1}, \dots, W_1 such that:

$$f = W_{J+1} \circ \sigma \bullet \dots \bullet \sigma \bullet W_1,$$

where, \bullet denotes component-wise composition. We use $\mathcal{NN}_{p,m}^\sigma$ to denote the set of DNNs of arbitrary width and depth. We always assume all activation functions to satisfy the following.

Assumption 1 (Kidger and Lyons (2020) Condition) *The activation function $\sigma \in C(\mathbb{R})$ is non-affine and there is a $x \in \mathbb{R}$ at which σ is differentiable. Moreover, $\sigma'(x) \neq 0$.*

2. Background

This section contains the metric-theoretic, topological, and Riemannian geometric background for the formulation of our results. Further background required only for proofs is relegated to the appendix.

2.1 Uniform Continuity

Since our approximation results are quantitative, focus on functions whose “metric distortion” is quantifiable; meaning that $f : \mathcal{X} \rightarrow \mathcal{Y}$ is continuous and its *optimal modulus of continuity*:

$$\omega(f, t) \triangleq \sup_{\substack{d_{\mathcal{X}}(x_1, x_2) \leq t \\ x_1, x_2 \in \mathcal{X}}} d_{\mathcal{Y}}(f(x_1), f(x_2)),$$

is finite for all $t \in [0, \infty)$. Such f are called *uniformly continuous* and the set of all uniformly continuous f mapping \mathcal{X} to \mathcal{Y} is denoted by $C(\mathcal{X}, \mathcal{Y})$. Many of our estimates require the inverse of f 's optimal modulus of continuity. However, even if ω is monotone increasing it need not be continuous and therefore in order to invert it we appeal to the *generalized inverse*, in the sense of Embrechts and Hofert (2013). This generalize inverse of $\omega(f, \cdot)$ is defined for $\epsilon > 0$ as follows:

$$\omega^{-1}(f, \epsilon) := \sup\{t : \omega(f, t) \leq \epsilon\}.$$

The notion of convergence on $C(\mathcal{X}, \mathcal{Y})$ is that of *uniform convergence on compact sets* inherited from the larger space of *continuous functions* from \mathcal{X} to \mathcal{Y} , denoted by $\bar{C}(\mathcal{X}, \mathcal{Y})$ ¹ and this notion of convergence in $\bar{C}(\mathcal{X}, \mathcal{Y})$ is defined via convergent sequences as follows. A sequence $\{f_n\}_{n=1}^\infty$ in $\bar{C}(\mathcal{X}, \mathcal{Y})$ converges *uniformly on compact sets* to some $f \in \bar{C}(\mathcal{X}, \mathcal{Y})$ if for every compact subset $K \subseteq \mathcal{X}$ and every $\epsilon > 0$, there is some positive integer N such that for any $n \geq N$:

$$\max_{x \in K} d_{\mathcal{Y}}(f_n(x), f(x)) < \epsilon.$$

Next, we discuss a few distinguished types of continuous functions relevant to our analysis.

2.2 Homeomorphisms and Homotopies

Topology studies geometric properties which are invariant up to continuous deformation. The strongest such notion is that of a *homeomorphism* ϕ from a metric space \mathcal{X} to a metric space \mathcal{Y} , which is a continuous bijection with continuous inverse. Effectively, since most topological properties are preserved either by continuous functions or by their inverses, then the existence of a homeomorphism $\phi : \mathcal{X} \rightarrow \mathcal{Y}$ implies that \mathcal{X} and \mathcal{Y} are topologically identical.

1. Many approximation results (e.g. Gühring et al. (2020b) or Kidger and Lyons (2020)) consider continuous functions on compact subsets of \mathbb{R}^p . In this situations, the Heine-Cantor Theorem ((Munkres, 2000, Theorem 27.6)) states every continuous function is uniformly continuous; i.e. $C(\mathcal{X}, \mathcal{Y}) = \bar{C}(\mathcal{X}, \mathcal{Y})$.

The existence of a homeomorphism between two metric spaces \mathcal{X} and \mathcal{Y} is a very strict condition implying that both spaces have many identical topological properties. Rather, both spaces can be considered as topologically similar if one can be “progressively deformed”. To formalize this idea, we need to define a *homotopy* between any two continuous functions $f, g : \mathcal{X} \rightarrow \mathcal{Y}$, which is a continuous function $F : [0, 1] \times \mathcal{X} \rightarrow \mathcal{Y}$ satisfying $F(0, x) = f(x)$ and $F(1, x) = g(x)$; here, $[0, 1] \times \mathcal{X}$ has the *product metric*, defined by

$$d_{[0,1] \times \mathcal{X}}((t_1, x_1), (t_2, x_2)) \triangleq \sqrt{|t_1 - t_2|^2 + d_{\mathcal{X}}(x_1, x_2)^2}.$$

We think of our two spaces as being topologically similar if there are continuous functions $f : \mathcal{X} \rightarrow \mathcal{Y}$ and $g : \mathcal{Y} \rightarrow \mathcal{X}$ for which there is a homotopy between $g \circ f$ and the identity on \mathcal{X} , as well as a homotopy between $f \circ g$ and the identity on \mathcal{Y} . NB, if a homeomorphism f between \mathcal{X} and \mathcal{Y} exists then we may take $g := f^{-1}$, $F(t, x) = x$ for all $(t, x) \in [0, 1] \times \mathcal{X}$ and $G(t, y) = y$ for all $(t, y) \in [0, 1] \times \mathcal{Y}$ to be the relevant homotopies.

Not all spaces which are homeomorphic are homotopic. In particular, the most relevant instance of this for this paper is the existence of a homotopy between a space and a point; or more generally, the existence of a homotopy between a continuous function $f : \mathcal{X} \rightarrow \mathcal{Y}$ and a constant function from \mathcal{X} to \mathcal{Y} . If f is homotopic to a constant function then we will say that f is said to be *null-homotopic*. In general, any two $f, g \in C(\mathcal{X}, \mathcal{Y})$ need not be homotopic; however, the situation simplifies drastically when the output space is Euclidean. NB, Euclidean spaces are precisely those relevant for most classical (uniform) universal approximation theorems (Hornik et al., 1989; Gühring et al., 2020b; Kidger and Lyons, 2020).

Example 1 *If \mathcal{X} is a normed-linear space then every $f \in C(\mathcal{X}, \mathbb{R}^m)$ is null-homotopic via the homotopy $(t, x) \mapsto (1 - t)f(x)$. In particular, every $f \in C(\mathbb{R}^p, \mathbb{R}^m)$ is null-homotopic.*

In contrast with Example 1, not all continuous functions are null-homotopic; for instance, it can be shown that the identity map of the circle is not null-homotopic. Thus, we can interpret null-homotopy as a formalization the idea that a function is “globally topologically simple”.

Homotopies allows us to define a key topological property, relevant to our analysis, called *simply connectedness*. A metric space \mathcal{X} is said to be simply connected if any pair of paths $\gamma_0, \gamma_1 : [0, 1] \rightarrow \mathcal{X}$ with the same endpoints, i.e: $\gamma_0(i) = \gamma_1(i)$ where $i = 0, 1$, there is a homotopy H from γ_1 to γ_2 which fixes the endpoints; i.e. $H(t, 0) = \gamma_0(t)$ and $H(t, 1) = \gamma_1(t)$ for all $t \in [0, 1]$. As in Example 1, all Euclidean spaces are simply connected, the circle is simply connected, but one can show that the Torus² is not.

2.3 Riemannian Geometry

Fix $p \in \mathbb{N}$. A (*p-dimensional*) *manifold* is a space which locally topologically resembles \mathbb{R}^p . More formally, a manifold is a topological space \mathcal{X} for which there is an *atlas* to \mathbb{R}^p ; i.e: a family $\{\phi_\alpha, U_\alpha\}_{\alpha \in A}$ of open subset $U_\alpha \subseteq \mathcal{X}$ with $\cup_\alpha U_\alpha = \mathcal{X}$ and homeomorphisms $\phi_\alpha : U_\alpha \rightarrow \mathbb{R}^p$. More broadly, a *manifold with boundary* refers to a topological space \mathcal{X} for which there is a collection $\{\phi_\alpha, U_\alpha\}_{\alpha \in A}$ (also called an atlas when clear from the context) of open subsets $U_\alpha \subseteq \mathcal{X}$ with $\cup_\alpha U_\alpha = \mathcal{X}$ and homeomorphisms ϕ_α from U_α to either \mathbb{R}^p or the “half-space” $\{(x_1, \dots, x_p) \in \mathbb{R}^p : x_p \geq 0\}$. Unless otherwise specified the term “manifold” will always refer to a manifold without boundary. We focus on manifolds whose geometry is locally comparable to \mathbb{R}^p , and not only their topology; i.e.: *Riemannian manifolds*.

Broadly speaking, a *p-dimensional complete Riemannian manifold* (without boundary) is a complete metric space \mathcal{X} , with metric $d_{\mathcal{X}}$, for which there are meaningful local notions of length, volume, curvature and differentiation, all of which are locally comparable to Euclidean space.

We will always assume that our Riemannian manifolds are complete, since this is a standard assumption made both when designing learning models of Riemannian manifolds (Hauberg Søren

2. The Torus is defined as \mathbb{R}^2 with the equivalence relation $(x, y) \sim (x + 1, y + 1)$ for each $(x, y) \in \mathbb{R}^2$ (see (Hatcher, 2002, page 46 and Proposition 1.6))

et al., 2013; Thomas Fletcher P., 2013; Schiratti et al., 2017) and when optimizing those models (Lezcano Casado, 2019; Ferreira et al., 2020). We impose geodesic completeness of our Riemannian manifolds, since amongst other things, it rules out pathological geometries such as $\mathbb{R}^2 - \{0\}$ with the Riemannian metric inherited from the Euclidean space \mathbb{R}^2 , wherein for example one cannot realize the distance between the points $(0, 1)$ and $(0, -1)$ with a distance-minimizing geodesic.

The local comparability happens on two fronts. The 0^{th} -order comparability requires that every $x \in \mathcal{X}$ be contained in some sufficiently small open ball $B_{\mathcal{X}}(x, \delta) \triangleq \{z \in \mathcal{X} : d_{\mathcal{X}}(z, x) < \delta\}$, for some $\delta > 0$, which can be mapped, via a smooth homeomorphism with smooth inverse, onto a sufficiently small Euclidean ball centered at 0 and of radius $\epsilon > 0$; we denote the latter by $B_{\mathbb{R}^p}(0, \epsilon) \triangleq \{z \in \mathbb{R}^p : \|z - 0\| < \epsilon\}$.

The first-order compatibility happens on the infinitesimal level by a set of copies of \mathbb{R}^m lying tangential to each $x \in \mathcal{X}$ called *tangent spaces*, each of which is denoted by $T_x(\mathcal{X})$. Each of these tangent spaces comes equipped with an inner product g_x , varying smoothly in x , which is used to formulate infinitesimal notions of angle and distance. Naturally, the 0^{th} and first-order comparability must be consistent and this happens when the distance $d_{\mathcal{X}}(x_1, x_2)$ between any two points $x_1, x_2 \in \mathcal{X}$ is realized by the *arc length* of an optimally efficient smooth path $\gamma : [0, 1] \rightarrow \mathcal{X}$ beginning at x_1 and ending at x_2 . Analogously to \mathbb{R}^m , the arc length of any such path is measured by $\int_0^1 \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} dt$; where $\dot{\gamma}(t)$ denotes the velocity vector at $\gamma(t)$. Any such path, called a *geodesic*, exists and is locally characterized as the unique solution to a particular *ordinary differential equation*, called the *geodesic equations* whose initial conditions determine the location and initial velocity of the geodesic. For any $x \in \mathcal{X}$, there corresponds a maximal Euclidean ball $B_{\mathbb{R}^p}(0, \text{inj}_{\mathcal{X}}(x))$ whose elements are all possible initial velocities to geodesics emanating from x and for which the map $\text{Exp}_{\mathcal{X}, x} : v \rightarrow \gamma(1)$ sending any initial velocity $v \in B_{\mathbb{R}^p}(0, \text{inj}_{\mathcal{X}}(x))$ to the point $\gamma(1)$, where γ is the geodesic beginning at x with initial velocity v is well-defined on the entire tangent space, and it is a homeomorphism near the origin. The quantity $\text{inj}_{\mathcal{X}}(x) \in [0, \infty]$ is called the *injectivity radius* at x and the map $\text{Exp}_{\mathcal{X}, x}$ is the *Riemannian exponential map* at x .

Suppose that $\dim(\mathcal{X}) > 1$. Given any $x \in \mathcal{X}$, consider an arbitrarily small triangle with vertex at x and whose sides are formed by geodesics emanating from x with initial velocities $v_1, v_2 \in T_x(\mathcal{X})$, and let $\pi_x(u, v)$ denote the 2-dimensional linear subspace of $T_x(\mathcal{X})$ spanned by v_1 and v_2 . The ratio of the gap between the sum of angles of that geodesic triangle with the sum of the angles of a triangle in Euclidean space $\pi_x(u, v)$, over the area of that geodesic triangle is a description of the curvature of \mathcal{X} at x . It is called the *sectional curvature* and denoted by $K_{\mathcal{X}}(\pi_x(u, v))$. We denote the set of all such smoothly varying tangent planes by $G_{p,2}(\mathcal{X})$. Similar methods can be used to define the *intrinsic volume* of any Borel subset $B \subseteq \mathcal{X}$, denoted by $\text{Vol}_{\mathcal{X}}(B)$. We say that a Riemannian manifold \mathcal{X} is orientable if it is impossible to smoothly move a two-dimensional figure along \mathcal{X} in such a way that the moving eventually results in the figure being flipped. Additional details surrounding Riemannian geometric can be found within the paper’s appendix.

3. Main Results on GDNs

In the remainder of this paper, we require that the geometries of the input and output spaces are “non-singular”; by which we mean that their curvature does not become unbounded and that the volume of any metric ball (of positive radius) never vanishes. Formally, we maintain the following.

Assumption 2 (Non-Degenerate Geometry: Cheeger et al. (1982))

There exist constants $v_{\mathcal{X}}, k_{\mathcal{X}} > 0$ satisfying:

- (i) $\sup_{\pi_x(u,v): x \in \mathcal{X}, \pi_x(u,v) \in G_{p,2}(\mathcal{X})} |K_{\mathcal{X}}(\pi_x(u, v))| \leq k_{\mathcal{X}}$,
- (ii) For any $\text{diam}(\mathcal{X}) > r > 0$, $\inf_{x \in \mathcal{X}} \text{Vol}_g(B_{\mathcal{X}}(x, r)) > 0$.

Moreover, *mutatis mutandis*, (i) and (ii) also hold for (\mathcal{Y}, h) .

Remark 2 *Pathological input or output space failing Assumptions 2, have been identified in the partner papers [Cheeger and Gromov \(1986\)](#), and [Cheeger and Gromov \(1990\)](#). However, it is difficult to imagine these constructions arising in practice.*

3.1 Differentiable Geometric Deep Learning is a Local Problem

Our first theoretical contribution is, to the best of our knowledge, the only known necessary condition for a function to be universally approximable (uniformly on compact sets). The result states that *any model class* $\mathcal{F} \subseteq C(\mathcal{X}, \mathcal{Y})$ is universal in $C(\mathcal{X}, \mathcal{Y})$ only if every function in $C(\mathcal{X}, \mathcal{Y})$ can be continuously deformed into some model in \mathcal{F} .

Lemma 3 (Deformability is Necessary for Universality) *Let $\mathcal{F} \subseteq C(\mathcal{X}, \mathcal{Y})$ and $f \in C(\mathcal{X}, \mathcal{Y})$. Then, for every $\epsilon > 0$ and every non-empty compact subset $K \subseteq \mathcal{X}$ there exists a $\hat{f} \in \mathcal{F}$ satisfying:*

$$\sup_{x \in K} d_{\mathcal{Y}}(f(x), \hat{f}(x)) < \epsilon,$$

only if: there exists an $H \in C([0, 1] \times \mathcal{X}, \mathcal{Y})$ and a model $\tilde{f} \in \mathcal{F}$ such that, for every $x \in \mathcal{X}$:

$$H(0, x) = f(x) \text{ and } H(1, x) = \tilde{f}(x). \tag{1}$$

In the non-Euclidean setting, universal approximation is faced with topological obstructions which are never present in the Euclidean setting of [Pinkus \(1999b\)](#) or [Kidger and Lyons \(2020\)](#), or in the more general \mathbb{R} -valued settings considered in [Chen et al. \(2018\)](#) and in [Yarotsky \(2021\)](#).

Example 2 (\mathbb{R}^m -Valued Maps are “Simple”) *The necessary condition for universality of Lemma 3 is always satisfied when considering \mathbb{R}^m -valued functions approximated by DNNs in $\mathcal{NN}_{p,m}^{\sigma}$ for any activation function $\sigma \in C(\mathbb{R})$. This is because, given any $\hat{f} \in \mathcal{NN}_{p,m}^{\sigma}$ and any target function $f \in C(K, \mathbb{R}^m)$, the following homotopy satisfies condition (1):*

$$H(t, x) \triangleq tf(x) + (1 - t)\hat{f}(x).$$

In contrast to Example 2, the behaviour of functions between even the simplest non-Euclidean geometries can be wildly complicated. For instance, there are infinitely many functions from the sphere to the circle which fail condition (1). Let $S^k \triangleq \{x \in \mathbb{R}^{k+1} : \|x\| = 1\}$.

Example 3 (Maps in Simple Non-Euclidean Manifolds are complicated) *There is a countably infinite family $\mathcal{F} \subseteq C(S^3, S^2)$ whose members can only approximate themselves, in the sense that: if $f_1, f_2 \in \mathcal{F}$ and $f_1 \neq f_2$ there does not exist an $H \in C([0, 1] \times S^3, S^2)$ satisfying $H(0, \cdot) = f_1$ and $H(1, \cdot) = f_2$. (The proof of this fact is in the paper’s appendix).*

Our first main result focuses on the observation that any model built from local data, in the sense of Figure 2, can only “globally approximate” if they are null-homotopic.

This necessary condition is summarized graphically in Figure 3, where we notice two “types of functions” on the sphere. The first is the identity function thereon (illustrated by the gray sphere itself), this is an example of a non-universally approximable target function. The second “type of function” is illustrated by each of the coloured paths, these functions are topologically defined by any model constructed from a “local interpretation of Figure 1” (formalized below) and they illustrate functions which can be universally approximated. Intuitively the difference between the gray function and the coloured functions is that the gray function can never be asymptotically deformed into one of the coloured functions without puncturing the sphere. I.e.: no such deformation as described by (1) in Lemma 3 is possible.

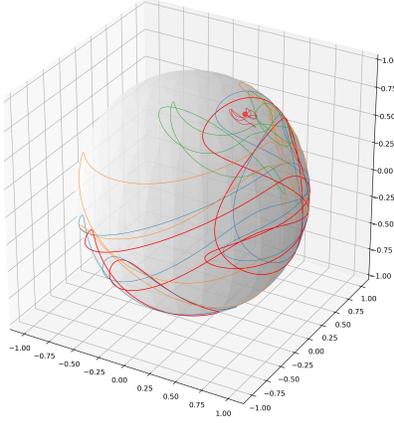


Figure 3: Visualizing the null homotopy-type condition of Theorem 6

We formalize the phrase “local interpretation of Figure 1”.

Definition 4 (Locally-Defined GDL Model) Fix atlases $\{\phi_\alpha, U_\alpha\}_{\alpha \in A}$ and $\{\rho_\zeta, V_\zeta\}_{\zeta \in Z}$ of \mathcal{X} and of \mathcal{Y} , respectively, and a “Euclidean model class” $\emptyset \neq \mathcal{F} \subseteq C(\mathbb{R}^p, \mathbb{R}^m)$. A locally defined GDL model is a family $\{\mathcal{F}_\alpha\}_{\alpha \in A}$ of models $\mathcal{F}_\alpha \subseteq C(\phi_\alpha(U_\alpha), \mathcal{Y})$, each of which is defined via:

$$\mathcal{F}_\alpha \triangleq \left\{ \rho_\zeta^{-1} \circ \hat{f} \circ \phi_\alpha : \hat{f} \in \mathcal{F}, \zeta \in Z \right\}. \quad (2)$$

Theorem 5 (Only Null-homotopic Maps are Approximable by Locally-Defined GDLs) Let \mathcal{X} and \mathcal{Y} be complete connected Riemannian manifolds of positive dimension, satisfying (2), and let $\{\mathcal{F}_\alpha\}_{\alpha \in A}$ be a locally-defined GDL model. For every $\alpha \in A$ and every compact $\emptyset \neq K \subseteq U_\alpha$:

- (i) For every $\zeta \in Z$ and every $g \in C(\mathbb{R}^p, \mathbb{R}^m)$, the map $\rho_\zeta^{-1} \circ g \circ \phi_\alpha$ is well-defined and in $C(K, \mathcal{Y})$,
- (ii) If there exists $f \in C(K, \mathcal{Y})$ which is not null-homotopic then, there is an $\epsilon > 0$ satisfying:

$$\inf_{\substack{g \in C(\mathbb{R}^p, \mathbb{R}^m) \\ \zeta \in Z, \alpha \in A}} \sup_{z \in K} d_{\mathcal{Y}} \left(\rho_\zeta^{-1} \circ g \circ \phi_\alpha(z), f(z) \right) \geq \epsilon.$$

Theorem 5 is a simple necessary condition for a map with non-Euclidean outputs to be globally approximable by feedforward networks. In particular, when the global geometry of \mathcal{Y} differs too greatly from Euclidean space, then functions which are not globally approximable necessarily *exist*.

Theorem 6 (Locally-Defined GDL Models are not Uniform Universal Approximators) Let \mathcal{X} and \mathcal{Y} be complete connected Riemannian manifolds, with $p \geq m$, \mathcal{Y} compact and orientable, \mathcal{X} satisfies Assumption 2, and let $\{\mathcal{F}_\alpha\}_{\alpha \in A}$ be a locally-defined GDL model. Then, there exists $\alpha \in A$, $\zeta \in Z$, a compact $\emptyset \neq K \subseteq \mathcal{X}$, $x \in K \subseteq U_\alpha$, an $\epsilon > 0$, and an $f \in C(K, \mathcal{Y})$ such that:

- (i) Each $\hat{f} \in \mathcal{F}_\alpha$ is a well-defined function in $C(K, \mathcal{Y})$,
- (ii) $\sup_{z \in K} \inf_{\hat{f} \in \mathcal{F}_\alpha} d_{\mathcal{Y}} \left(\hat{f}(z), f(z) \right) \geq \epsilon.$

3.1.1 DISCUSSION: WHY UNIFORM APPROXIMATION POORLY SUITED TO GDL PROBLEMS

The necessary condition for universality identified in Lemma 3 causes a major obstruction to building universal approximators in $C(\mathcal{X}, \mathcal{Y})$. This is because, the model class $\mathcal{F} \subset C(\mathcal{X}, \mathcal{Y})$ needs to exhaust

all the homotopy types therein. However, verifying that this condition is met is at-least as difficult as computing the homotopy groups of the output space \mathcal{Y} (see [Fomenko and Fuchs \(2016\)](#) for details), which has recently been shown in [Čadek et al. \(2014\)](#) and in [Matousek \(2013\)](#) to be an (at-least) *NP-hard* problem. Therefore, verifying the compatibility of any model class $\mathcal{F} \subset C(\mathcal{X}, \mathcal{Y})$ with the geometry of \mathcal{Y} is *computationally infeasible*.

In the simplified setting where one instead considers only locally defined GDL models, [Theorem 6](#) guarantees that when \mathcal{X} and \mathcal{Y} are both compact and connected Riemannian manifolds of positive dimension then there are functions in $C(\mathcal{X}, \mathcal{Y})$ which cannot approximate all functions in $C(\mathcal{X}, \mathcal{Y})$ uniformly on *arbitrarily large compact subsets of \mathcal{X}* . Thus, any locally-defined GLD model is faced with the following problem: either the conditions for [Theorem 6](#) are met, and therefore, the model class is not universal, or it is computationally infeasible to verify if the model class is universal.

Therefore, uniform approximation on “uncontrolled compact subsets” (i.e. of arbitrarily large maximum diameter) of a function between general Riemannian manifolds is not a well-suited notion of “universal approximation” for geometric deep learning. However, as we will now show, all these obstructions vanish when the models are only required to approximate the target function on compact subsets of \mathcal{X} with a certain maximum diameter.

We now introduce the notion of “*controlled universal approximation*” (i.e.: universal approximation on compact subsets with a specific bounded maximum diameter). Moreover, we find that our GDN models are universal in this sense. We show that controlled universal approximation coincides with uniform approximation on compact sets when \mathcal{X} and \mathcal{Y} are non-positively curved (e.g. Euclidean space). Therefore, this notion of universality strictly extends the familiar notion of [Hornik \(1991\)](#), [Pinkus \(1999b\)](#), and [Kidger and Lyons \(2020\)](#) to the non-Euclidean setting without any of the topological obstructions of the “naive” uniform approximation on compact sets notion of universality.

Remark 7 (Connection to Relative Forms of Uniform Convergence) *In the general case, where \mathcal{X} or \mathcal{Y} may have somewhere positive curvature (e.g. any compact Riemannian manifolds of positive dimension) our notion of “controlled universality approximation” is most similar to density in the relative uniform convergence topologies introduced in [Arens and Dugundji \(1951\)](#) and studied in [McCoy and Ntantu \(1988\)](#), [Nokhrin and Osipov \(2009\)](#), and in [Bouchair and Kelaiaia \(2014\)](#).*

3.2 Controlled Universal Approximation

We also make use of the function sending any $x \in \mathcal{X}$ and any $K \in (0, \infty]$ to:

$$\delta(\mathcal{X}, x, K) \triangleq \sup_{0 < r < K} r \frac{\text{Vol}_{\mathcal{X}}(B_{\mathcal{X}}(x, r))}{\text{Vol}_{\mathcal{X}}(B_{\mathcal{X}}(x, r)) + \text{Vol}_{T_x(\mathcal{X})}(B_{T_x(\mathcal{X})}(0, 2r))};$$

note, that $\delta(\mathcal{Y}, y, K)$ is defined analogously. Our analysis relies on the following function, mapping any $K \in \mathbb{R}$ to the extended-real number:

$$K^* \triangleq \begin{cases} \frac{\pi}{4\sqrt{K}} & : K > 0 \\ \infty & : K \leq 0. \end{cases}$$

Our approximation results concern the following locally-defined GDL model.

Definition 8 (Geometric Deep Feedforward Networks) *Fix $\sigma \in C(\mathbb{R})$. A geometric deep feedforward network (GDN) from \mathcal{X} to \mathcal{Y} at $x \in \mathcal{X}$ with activation function σ , is a function $\hat{f} \in C(B_{\mathcal{X}}(x, \text{inj}_{\mathcal{X}}(x)), \mathcal{Y})$ with representation:*

$$\hat{f} \triangleq \text{Exp}_{\mathcal{Y}, y} \circ g \circ \text{Exp}_{\mathcal{X}, x}^{-1},$$

for some $g \in \mathcal{NN}_{p, m}^{\sigma}$ and some $y \in \mathcal{Y}$.

Theorem 9 (Controlled Universal Approximation) *Let \mathcal{X} and \mathcal{Y} be connected complete Riemannian manifolds satisfying Assumption 2, of respective dimensions p and m , suppose that \mathcal{X} is compact, and let σ be an activation function satisfying Assumption 1. For any continuous function $f : \mathcal{X} \rightarrow \mathcal{Y}$, any $\epsilon > 0$, and any $x \in \mathcal{X}$, if:*

$$\delta < \min \{ \text{inj}_{\mathcal{X}}(x), \omega^{-1}(f, \text{inj}_{\mathcal{Y}}(f(x))) \}, \quad (3)$$

then the following hold:

(i) **Well-Definedness of GDN:** *For every $g \in \mathcal{NN}_{p,m,p+m+2}^{\sigma}$ the map $\hat{f} \triangleq \text{Exp}_{\mathcal{Y},f(x)} \circ g \circ \text{Exp}_{\mathcal{X},x}^{-1}$ is well-defined on $\overline{B_{\mathcal{X}}(x,\delta)}$,*

(ii) **Controlled Universal Approximation:** *There is a GDN \hat{f} as in (i) satisfying:*

$$\sup_{\tilde{x} \in \overline{B_{\mathcal{X}}(x,\delta)}} d_{\mathcal{Y}}(f(\tilde{x}), \hat{f}(\tilde{x})) \leq \epsilon.$$

(iii) **GDN Complexity Estimate:** *The depth of g is recorded in Table 1, and it depends on σ 's regularity.*

Furthermore, the right-hand side of (3) is lower-bounded via:

$$\min \{ \delta(\mathcal{X}, x, k_{\mathcal{X}}^*), \omega^{-1}(f, \delta(\mathcal{Y}, f(x), k_{\mathcal{Y}}^*)) \} \leq \min \{ \text{inj}_{\mathcal{X}}(x), \omega^{-1}(f, \text{inj}_{\mathcal{Y}}(f(x))) \}. \quad (4)$$

Regularity of σ	Order of Depth
$C^{\infty}(\mathbb{R}) + \text{Non-polynomial}$	$O\left(\frac{m(2\delta)^{2p}}{\kappa_2^{2p}\left(\omega^{-1}\left(f, \frac{\epsilon\kappa_1}{(1+\frac{p}{4})m}\right)\right)^{2p}}\right)$
Non-affine polynomial ³	$O\left(\frac{m(m+p)(2\delta)^{4p+2}}{\kappa_2^{4p+2}\left(\omega^{-1}\left(f, \frac{\epsilon\kappa_1}{(1+\frac{p}{4})m}\right)\right)^{4p+2}}\right)$
$C(\mathbb{R}) + \text{Non-polynomial}$	$O\left(\frac{m(2\delta)^{2p}}{\kappa_2^{2p}\omega^{-1}\left(f, \frac{\epsilon\kappa_1}{2m(1+\frac{p}{4})}\right)^{2p}\left(\kappa_2\omega^{-1}\left(\sigma, \frac{\epsilon}{(2\delta)^2[\omega^{-1}\left(f, \frac{\epsilon\kappa_1}{2m(1+\frac{p}{4})}\right)]^{-2+1}}\right)}\right)}\right)$

Where $\kappa_1, \kappa_2 >$ depend only on the curvature of \mathcal{X} at x and of \mathcal{Y} at $f(x)$, respectively.

Table 1: Approximation rates for GDNs based on activation function

Theorem 9 guarantees that universal approximation by GDNs on compact subsets of general Riemannian manifolds whose size is “controlled by the right-hand side of (3)” is possible; even if Theorem 6 mandates it typically fails “globally”; i.e., for arbitrarily large compact subsets of \mathcal{X} . Thus, the “radius” in (3) quantifies the *gap between “local” and “global” universal approximation*.

Definition 10 (Universality Radius) *Let \mathcal{X} and \mathcal{Y} be Riemannian manifolds, and let $f \in C(\mathcal{X}, \mathcal{Y})$. The universality radius of f at any $x \in \mathcal{X}$ is defined to be the quantity:*

$$\mathcal{U}_f(x) \triangleq \min \{ \text{inj}_{\mathcal{X}}(x), \omega^{-1}(f, \text{inj}_{\mathcal{Y}}(f(x))) \}.$$

Remark 11 (Analogy: Taylor Expansions and Controlled Universal Approximation) *The universality radius of any $f \in C(\mathcal{X}, \mathcal{Y})$ plays a similar role to the radius and interval of convergence*

of a smooth function in classical calculus on \mathbb{R} . This is because, on the interval of convergence about any $x \in \mathcal{X}$ a smooth function $f \in C^\infty(\mathbb{R}, \mathbb{R})$ can be locally approximated to arbitrary precision by its Taylor series. Analogously, any $f \in C(\mathcal{X}, \mathcal{Y})$ can be universally approximated by a GDN on $B_{\mathcal{X}}(x, \mathcal{U}_f(x))$. In both cases, the radius depends on the point of the input space about which the approximation is performed and on the regularity of the function.

One may ask if there is a broad class of input/output spaces for which the obstruction of Theorem 6 vanishes. In such cases, the GDN architecture can be developed about any point of the input space with the confidence that the lower-bound (4) is infinity.

3.2.1 LOCAL-TO-GLOBAL UNIVERSALITY FOR CARTAN-HADAMARD MANIFOLDS

Our search for input or output spaces with generically infinite universality radii begins with the Cartan-Hadamard Theorem (see (Jost, 2017, Corollary 6.9.1)) and *Cartan-Hadamard* manifolds. These are simply connected, complete Riemannian manifolds of everywhere non-positive sectional curvature. Three important examples in geometric deep learning are the Hyperbolic spaces, the manifold of non-degenerate *Gaussian probability measures* with the Fisher-Rao distance (from information geometry; see (Ay et al., 2017, Equation 3.22)), and the familiar Euclidean spaces.

For Cartan-Hadamard manifolds, we have the following “local-to-global” result. That is, the next result describes a broad range of situations in which controlled universal approximation coincides with density in the uniform convergence on compact sets topology on $C(\mathcal{X}, \mathcal{Y})$.

Corollary 12 (From Local to Global Universal Approximation) *If \mathcal{X} and \mathcal{Y} are Cartan-Hadamard manifolds, then the following estimate holds:*

$$\inf_{x \in \mathcal{X}} \inf_{f \in C(\mathcal{X}, \mathcal{Y})} \mathcal{U}_f(x) = \infty.$$

3.3 Breaking the Curse of Dimensionality via Efficient Datasets

3.3.1 DISCUSSION: OVERVIEW OF OUR APPROACH

Thus far, as in most universal approximation papers, the objective has been to approximate $f \in C(\mathcal{X}, \mathcal{Y})$ on arbitrary compact subsets of \mathcal{X} for which universal approximation is not obstructed by Theorem 6. Indeed, classical constructive approximation results found in DeVore and Lorentz (1993) guarantee that cursed approximation rates (as in Theorem 9) are unavoidable. This phenomenon can be equally seen in the simple Euclidean case where it is confirmed in Gühring et al. (2020b) that the best possible approximation rates for feedforward networks with ReLU activation function are unavoidably exponential in the involved spatial dimensions and the approximation error. Thus, the universal approximation problem is “cursed from the start” since we looked for a general rate which applies to any uniformly continuous function on any compact subset of the input space.

As pioneered in the quantitative approximation theorem of Barron (1993), the author found that the curse of dimensionality can be avoided if restrictions are placed on the set of functions which are considered for approximated. Since then, several other authors; e.g. Barron (1993), Yarotsky and Zhevnerchuk (2020), Siegel and Xu (2020), Gühring et al. (2020a), Suzuki (2019), and Cheridito et al. (2021), have identified sub classes of function which can be approximated by DNNs whose number of parameters does not depend adversely on the dimension of the input and output spaces⁴ (potentially in different function spaces). We highlight that, each of these results takes a “*functioncentric* perspective” in that they focus on the impact of f ’s regularity on the neural network approximation rates and omit the impact of the dataset on these approximation rates.

4. These approximation results are not all in $C(\mathbb{R}^p, \mathbb{R}^m)$ for the uniform distance; nevertheless, they all have the commonality of avoiding the curse of dimensionality by restricting the class of approximated functions within some larger function space in which the DNNs are dense.

Here, we instead consider a “*datacentric perspective*” wherein we ask: *given an $f : \mathcal{X} \rightarrow \mathcal{Y}$, on which datasets⁵ \mathbb{X} (i.e.: non-empty subsets of \mathcal{X}) can f be uniformly approximated by a GDN whose number of parameters does not depend adversely on the dimension of \mathcal{X} and of \mathcal{Y} ?* Such datasets will be called *efficient for f* . We will see that, if f is sufficiently smooth then any dataset \mathbb{X} is efficient for f and, conversely, every “real-world dataset” (i.e. finite and non-empty) \mathbb{X} is efficient for any (potentially discontinuous) function $f : \mathcal{X} \rightarrow \mathcal{Y}$.

Remark 13 (Implications in the Euclidean Case) *In particular, as developed further in Section 4.5, our result strictly extend the dimension-free rates for DNNs in $C([0, 1]^p, \mathbb{R})$ obtained recently in Yarotsky and Zhevnerchuk (2020). Hence, even in the Euclidean case, our datacentric perspective is both novel and more general than the functioncentric perspective.*

The idea of our approach is concisely summarized by Figure 4 wherein see that the green function F coincides with the **target function f** on the **dataset \mathbb{X}** but F is much more regular. Thus, if we instead approximate the more regular function F by a GDN on all of the input space, then we can do so with a GDN which avoids the curse of dimensionality and simultaneously obtain an equally accurate approximation of **target function f** on **dataset \mathbb{X}** since both **target function f** and the green function F coincide thereon. Note that, our next main results does not assume that \mathbb{X} is finite.



Figure 4: What are efficient datasets?

To formalize our task, we need to define what we mean by F being “regular enough”. We consider an extension of the notion of regularity studied in Yarotsky and Zhevnerchuk (2020) but in the Riemannian context. Following Jost (2017), we make the following definition. We say that a function $f \in \bar{C}(\mathcal{X}, \mathcal{Y})$ is regular if it has many higher-order partial derivatives, and if the last one of which locally distorts distance up to a linear scaling factor.

Definition 14 ($C_{loc}^{k,1}(\mathcal{X}, \mathcal{Y})$) *Fix a $k \in \mathbb{N}_+$, and fix smooth atlases $(\phi_\alpha, U_\alpha)_{\alpha \in A}$ and $(\psi_\zeta, V_\zeta)_{\zeta \in Z}$ of \mathcal{X} and of \mathcal{Y} respectively. We say that $f \in \bar{C}(\mathcal{X}, \mathcal{Y})$ belongs to $C_{loc}^{k,1}(\mathcal{X}, \mathcal{Y})$ if, for every $\alpha \in A$ and every non-empty compact subset $K \subseteq \phi_\alpha(U_\alpha)$ we have:*

$$\max_{\beta: |\beta| \leq k} \max_{x \in K} \|D^\beta \pi_i \circ \phi_\alpha \circ f \circ \psi_\zeta^{-1}(x)\| + \sup_{x, y \in K, x \neq y} \frac{\|D^\beta \pi_i \circ \phi_\alpha \circ f \circ \psi_\zeta^{-1}(x) - D^\beta \pi_i \circ \phi_\alpha \circ f \circ \psi_\zeta^{-1}(y)\|}{d_Y(x, y)} < \infty;$$

5. Note a dataset need not be a training dataset but rather refers to the set on which we expect our approximation to hold.

whenever the composition $\phi_\alpha \circ f \circ \psi_\zeta^{-1}$ is well-defined and where $\pi_i : \mathbb{R}^m \ni x \mapsto x_i \in \mathbb{R}$ is the canonical projection onto the i^{th} -coordinate, $D^\beta = \frac{\partial^{|\beta|}}{\partial x_{\beta_1} \dots \partial x_{\beta_{|\beta|}}}$, and $|\beta|$ is the length of the multi-index β .

Therefore, our approach will be the following: replace the target function f in \mathcal{X} by a sufficiently smooth $F : \mathcal{X} \rightarrow \mathcal{Y}$. Here, sufficiently smooth means that F admits all k continuous partial derivatives for some integer k divisible by the dimension p of \mathcal{X} ; that is $F \in C_{\text{loc}}^{k,1}(\mathcal{X}, \mathcal{Y})$ and $k = np$ for some positive integer n . In this case, we may approximate F by a GDN depending on few parameters over a “regular” subset of \mathcal{X} containing \mathbb{X} and then restrict our approximation of F to \mathbb{X} thereby efficiently approximating f . In what follows, we will denote the cardinality of a dataset \mathbb{X} by $\#\mathbb{X}$.

Remark 15 (The roles of k and of n) Suppose that \mathbb{X} is finite. If n equals to the number of points in \mathbb{X} (i.e. if $n = \#\mathbb{X}$) then we would be seeking a function $F \in C_{\text{loc}}^{np,1}(\mathcal{X}, \mathcal{Y})$ satisfying

$$F(x) = f(x) \quad (\text{for all } x \in \mathbb{X}). \quad (5)$$

Under the conditions that $k = \#\mathbb{X}p$, the “smoothness” of f ’s extension F on the dataset \mathbb{X} is effectively coupled with the dimension p of \mathcal{X} but it is also coupled to the number of datapoints in \mathbb{X} . In this case, our next result (Theorem 20) implies that for any $\epsilon > 0$, f can be approximated to ϵ -precision on \mathbb{X} by a GDN determined by $\mathcal{O}(m(m^2 - 1)\epsilon^{-2p/3(np+1)})$ trainable parameters.

However, the requirement that $k = \#\mathbb{X}p$ places a heavy restriction on the candidate smooth functions F which could satisfying (5), since the condition $k = \#\mathbb{X}p$ necessitates that F must be very smooth whenever $\#\mathbb{X}$ is large. A fortiori, this formulation is meaningless for any infinite \mathbb{X} .

In fact, for our efficient approximation result (Theorem 20) to hold we do not need that $k = \#\mathbb{X}p$; rather, we only that $F \in C_{\text{loc}}^{k,1}(\mathcal{X}, \mathcal{Y})$ where k is some positive integer divisible by p . Therefore, by prespecifying some positive integer (n) such that $k = np$ and looking for an $F \in C_{\text{loc}}^{k,1}(\mathcal{X}, \mathcal{Y})$ satisfying (5) we may still conclude that f can be approximated on \mathbb{X} to ϵ -precision by a GDN determined by $\mathcal{O}(m(m^2 - 1)\epsilon^{-2p/3(np+1)})$ parameters. Moreover, by decoupling k from the cardinality of \mathbb{X} in this way, we no longer constrain the collection of “candidate functions” F satisfying (5) for large (but finite) dataset \mathbb{X} . Furthermore, by decoupling k from $\#\mathbb{X}$ we can also meaningfully handle the case where \mathbb{X} is infinite.

When this is possible, we show that the geometric arguments of Theorem 9 may be combined with an extension of the recently efficient approximation results of Yarotsky and Zhevnerchuk (2020) (describing efficient approximation of functions in $C_{\text{loc}}^{np,1}(\mathbb{R}^p, \mathbb{R})$ by models in $\mathcal{NN}_{p,1}^{\text{ReLU}}$; where, $\text{ReLU} : x \rightarrow \max\{0, x\}$) to obtain an efficient approximating of f by GDNs. The problem of replacing a function f by a $C_{\text{loc}}^{np,1}(\mathcal{X}, \mathcal{Y})$ function coinciding with it on \mathbb{X} is equivalent to the problem of extending f on \mathbb{X} to such a function. This latter problem is known as the *Whitney Extension Problem*, and dates back to Whitney (1934)⁶. Fortunately, this long-standing open problem has recently been solved in a series of papers: Bierstone et al. (2003), Fefferman (2005), and Bierstone et al. (2006). We leverage these analytic results to solve our efficient universal approximation problem⁷.

3.3.2 EFFICIENT DATASETS

Our analysis begins by reformulating the conditions of Fefferman (2005)’s Whitney Extension Theorem to suit our controlled universal approximation context. The best known conditions, to the authors’ knowledge, are (in the language of our context) the following.

6. We require Fefferman (2005)’s Extension Theorem since we are interested in *uniform-type approximation results*. If one were interested in applying our approach to other notions of approximation, e.g. Sobolev, or Besov norms, then the recent development surrounding extension theorems; see Fefferman et al. (2014), Heikkinen et al. (2016), Ambrosio and Puglisi (2020), or Bruè et al. (2021), would likely be equally central to obtaining “datacentric” uncursed rates by GDNs (or even classical DNNs) in those contexts.
7. A proper treatment of the Whitney Extension Problem is not aligned with our paper’s length target. The interested reader is referred to: Brudnyi and Brudnyi (2012a,b).

Definition 16 (Efficient Datasets) Fix $n \in \mathbb{N}_+$, an $f : \mathcal{X} \rightarrow \mathcal{Y}$, a dataset \mathbb{X} , and set:

$$\ln(C^*) \triangleq \min\{\ln(\#\mathbb{X}), 2^C \ln(C+1)\} \text{ and } C \triangleq \binom{p+np}{p}.$$

Then, \mathbb{X} is n -efficient for f at $x \in \mathcal{X}$ if the following holds: for each $\{x_c\}_{c=1}^{C^*} \subseteq \mathbb{X}$ there exists an $M > 0$, independent of $\{x_c\}_{c=1}^{C^*}$, and polynomials $p_c \in C(\mathbb{R}^p, \mathbb{R}^m)$ of degree np satisfying:

- (i) $p_c(\text{Exp}_{\mathcal{X},x}^{-1}(x_c)) = \text{Exp}_{\mathcal{Y},f(x)}^{-1}(f(x_c))$, for all $c = 1, \dots, C^*$
- (ii) $|\partial^\beta p_c(\text{Exp}_{\mathcal{X},x}^{-1}(x_c))_i| \leq M$, for all $c = 1, \dots, K$, $|\beta| \leq np$, and $i = 1, \dots, p$,
- (iii) $|\partial^\beta (p_c - p_j)(\text{Exp}_{\mathcal{X},x}^{-1}(x_c))_i| \leq M \left\| \text{Exp}_{\mathcal{X},x}^{-1}(x_c) - \text{Exp}_{\mathcal{X},x}^{-1}(x_j) \right\|^{nd-|\beta|}$, for all $c, j = 1, \dots, C^*$, $|\beta| \leq np$, and $i = 1, \dots, p$,

where $p_c(z)_i$ denotes the projection of p_c onto its i^{th} coordinate evaluated at z . We say that \mathbb{X} is n -efficient for f if it is n -efficient for f at each $x \in \mathcal{X}$.

We begin by showing that functions for which all datasets are efficient extend the class of efficiently approximable functions of [Yarotsky and Zhevnerchuk \(2020\)](#) to the general Riemannian case.

Proposition 17 (Every Dataset is Efficient for $C_{\text{loc}}^{np,1}(\mathcal{X}, \mathcal{Y})$ -Functions) Fix $f \in C_{\text{loc}}^{np,1}(\mathcal{X}, \mathcal{Y})$ and let \mathbb{X} be a dataset satisfying the following: there is an $0 \leq \eta < 1$ and a $x \in \mathcal{X}$ such that:

$$\mathbb{X} \subseteq B_{\mathcal{X}}(x, \eta \text{inj}_{\mathcal{X}}(x)) \quad \text{and} \quad f(\mathbb{X}) \subseteq B_{\mathcal{Y}}(f(x), \eta \text{inj}_{\mathcal{Y}}(f(x))); \quad (6)$$

then, $\mathbb{X} \subseteq \mathcal{X}$ is n -efficient for f . In particular, condition (6) always holds if both \mathcal{X} and \mathcal{Y} are Cartan-Hadamard manifolds.

Proposition 17 is doubly insightful since it implies that functions for which every dataset is efficient are typical, from the approximation-theoretic standpoint.

Corollary 18 (Functions for Which Every Dataset is Efficient are Generic) Consider the setting of Proposition 17. Let $C^{\text{eff}}(\mathcal{X}, \mathcal{Y})$ denote the set of all $f \in \bar{C}(\mathcal{X}, \mathcal{Y})$ with the following property: for every $x \in \mathcal{X}$, each $0 \leq \eta < 1$, and every finite $\mathbb{X} \subseteq B_{\mathcal{X}}(x, \eta \mathcal{U}_f(x))$, there is some positive integer n for which \mathbb{X} is n -efficient for f . Then, the set $C^{\text{eff}}(\mathcal{X}, \mathcal{Y})$ is dense in $\bar{C}(\mathcal{X}, \mathcal{Y})$.

Conversely, datasets for which every function is “efficient” are also prevalent. In fact, every “real-world dataset” (i.e. a non-empty finite dataset) has this property.

Proposition 19 (Real-World Datasets are Efficient for Any Function) Let \mathbb{X} be a finite set and $f : \mathcal{X} \rightarrow \mathcal{Y}$. Suppose that $\mathbb{X} \subset B_{\mathcal{X}}(x, \eta \mathcal{U}_f(x))$, for some $x \in \mathcal{X}$ and some $0 < \eta < 1$. Assume that $\#\mathbb{X} < p$ and that p divides $\#\mathbb{X} - 1$. Then \mathbb{X} is $\frac{\#\mathbb{X}-1}{p}$ -efficient for f .

Together, Propositions 17 and 19 show that efficient datasets describe a rich host of situations which are well beyond the scope of the classical perspective of assuming additional regularity of f . To ensure a consistent narrative with the recent developments in [Gühring et al. \(2020b\)](#), [Yarotsky \(2017\)](#), and [Yarotsky and Zhevnerchuk \(2020\)](#), we focus on normalized datasets.

Assumption 3 (Normalizable Dataset) Let $\mathbb{X} \subseteq \mathcal{X}$ be a dataset and $f : \mathcal{X} \rightarrow \mathcal{Y}$. Then, \mathbb{X} is f -normalizable if there is some $x \in \mathcal{X}$ and some $0 < \eta < 1$ such that $\mathbb{X} \subset B_{\mathcal{X}}(x, \eta \mathcal{U}_f(x))$ and

$$\text{Exp}_{\mathcal{X},x}^{-1}(\mathbb{X}) \subset [0, 1]^p; \quad (7)$$

Example 4 (Normalizability in the Euclidean Setting) If $\mathcal{X} = \mathbb{R}^p$ and $\mathcal{Y} = \mathbb{R}^m$ then, $\mathcal{U}_f(x) = \infty$ and $\text{Exp}_{\mathbb{R}^p,x}(y) = y + x$; thus, condition (7) reduces to $\{z - x\}_{z \in \mathbb{X}} \subseteq [0, 1]^p$.

3.3.3 BREAKING THE CURSE OF DIMENSIONALITY ON EFFICIENT DATASETS

Our result focuses on *piecewise linear* activation functions. By a piecewise linear activation function, we mean a $\sigma \in C(\mathbb{R})$ for which there exists $B \in \mathbb{N}$ and distinct $x_1, \dots, x_B \in \mathbb{R}$ satisfying: every $x \in \mathbb{R} - \{x_b\}_{b=1}^B$ is contained in an open interval in which σ is linear and there is no such interval for each x_b (for $b = 1, \dots, B$). Note, if σ is piecewise linear and non-affine then $B \geq 1$. These include the ReLU activation function of Fukushima (1969), the leaky-ReLU activation function of Maas et al. (2013), the pReLU activation function of He et al. (2015), commonly implemented piecewise linear approximations to the Heavyside function (implemented for example in Abadi et al. (2015) and in Team et al. (2016)), and many others.

Let $g \in \mathcal{NN}_{p,m}^\sigma$ have representation $g = W_J \circ \sigma \bullet \dots \bullet \sigma \bullet W_1$ and $W_j(x) \triangleq A_j x + b_j$ (where A_j is a $p_{j+1} \times p_j$ matrix, $b_j \in \mathbb{R}^{p_j}$, $p_1 = p$, and $p_J = m$) for some $J \in \mathbb{N}_+$ with $J > 1$. Following Cheridito et al. (2021), the total number of *trainable parameters* in this representation of g is defined by:

$$\sum_{j=1}^J p_j(p_{j-1} + 1).$$

Theorem 20 (Polynomial Approximation Rates On Efficient Datasets) Fix $n \in \mathbb{N}_+$, let $f : \mathcal{X} \rightarrow \mathcal{Y}$, σ be a non-affine piecewise linear activation function, and let \mathbb{X} be an f -normalizable and n -efficient dataset for f . Then, for each $\epsilon > 0$, there is a $W \in \mathbb{N}_+$, a $g \in \mathcal{NN}_{p,m,W}^\sigma$, and a constant $\kappa > 0$ (not depending on ϵ , p , or on m), such that the GDN: $\hat{f} \triangleq \text{Exp}_{\mathcal{Y},f(x)} \circ g \circ \text{Exp}_{\mathcal{X},x}^{-1}$, satisfies the uniform estimate:

$$\sup_{x \in \mathbb{X}} d_{\mathcal{Y}} \left(f(x), \hat{f}(x) \right) \leq \kappa m^{\frac{1}{3}} \epsilon. \quad (8)$$

Moreover, g satisfies the following sub-exponential complexity estimates:

- (i) **Width:** satisfies $m \leq W \leq m(4p + 10)$,
- (ii) **Depth:** of order $\mathcal{O} \left(m + m \epsilon^{\frac{2p}{3(np+1)} - \frac{p}{np+1}} \right)$,
- (iii) **Number of trainable parameters:** is of order $\mathcal{O} \left(m(m^2 - 1) \epsilon^{-\frac{2p}{3(np+1)}} \right)$.

Remark 21 (Dimension-Free Rates) If \mathbb{X} is 1-efficient for f , then the network g of Theorem 20 has depth roughly of the order $\mathcal{O} \left(m + m \epsilon^{\frac{-1}{3}} \right)$ and it depends on $\approx \mathcal{O} \left(m^2 \epsilon^{-\frac{2}{3}} \right)$ trainable parameters.

Remark 22 (Discussion: Efficiency Datasets Vs. Target Functions Regularity) An advantage of our efficient dataset approach to “non-cursed” approximation rates over the classical approach, which imposes regularity assumptions on the target function, is a practical one. Namely, given any dataset \mathbb{X} , the Definition 16 can directly be verified. However, any additionally assumed regularity of the target function typically cannot be verified in practice.

3.4 Applications

We illustrate our theoretical framework developed thus far by establishing the universality of many commonly deployed geometric deep learning models.

3.4.1 HYPERBOLIC FEEDFORWARD NETWORKS ARE UNIVERSAL

Hyperbolic spaces have gained significant recent interest, in geometric deep learning, for their ability to represent complex tree-like structures much more efficiently and faithfully than Euclidean representations. Examples of such state-of-the-art embeddings include low-dimension representations of

complex hierarchical datasets used in Nickel and Kiela (2017), efficient representations of complex social networks in Krioukov et al. (2010), tractable representations of large undirected graphs in Munzner (1997), and accurate representations of trees in Sala et al. (2018). Accordingly, the hyperbolic feedforward networks of Ganea et al. (2018), and Shimizu et al. (2021) have gained significant recent interest due to their ability to process such representations since they have inputs and outputs in (generalized) hyperbolic spaces. Let us briefly recall these notions before establishing the relevant quantitative universal approximation guarantees.

The (generalized) hyperbolic spaces \mathbb{D}_c^n is the Cartan-Hadamard manifold $\{x \in \mathbb{R}^n : c\|x\|^2 < 1\}$ whose Riemannian structure induces the distance function:

$$d_c(x, y) \triangleq \frac{2}{\sqrt{c}} \tanh^{-1} \left(\sqrt{c} \left\| \frac{(1 - c\|x\|^2)y - (1 - 2cx^\top y + c\|y\|^2)}{1 - 2cx^\top y + c^2\|x\|^2\|y\|^2} \right\| \right).$$

The hyperbolic feedforward networks of Ganea et al. (2018) are defined via a series of complicated operations; however, as the authors later note (Ganea et al., 2018, Equation (26)) every hyperbolic feedforward network $\hat{f} : \mathbb{D}_c^p \rightarrow \mathbb{D}_c^m$ can equivalently be represented by:

$$\hat{f} = \text{Exp}_{\mathbb{D}_c^p, 0} \circ f \circ \text{Exp}_{\mathbb{D}_c^m, 0}^{-1}, \quad (9)$$

where $f \in \mathcal{NN}_{p,m}^\sigma$. We note that closed-form expressions for $\text{Exp}_{\mathbb{D}_c^m, 0}$ and $\text{Exp}_{\mathbb{D}_c^m, 0}^{-1}$ are known (see (Ganea et al., 2018, Lemma 2)). Our framework therefore implies the following universal approximation theorem for hyperbolic feedforward networks, which is a quantitative version of (Kratsios and Bilokopytov, 2020, Corollary 3.16).

Corollary 23 (Hyperbolic Neural Networks are Universal Approximators) *Let $\sigma \in C(\mathbb{R})$ satisfy the Kidger-Lyons conditions. Fix $c, \epsilon > 0$, $f \in C(\mathbb{D}_c^p, \mathbb{D}_c^m)$, and a non-empty compact subset $K \subseteq \mathbb{D}_c^p$. Then, there exists a hyperbolic feedforward network \hat{f} satisfying:*

$$\sup_{x \in K} d_c(f(x), \hat{f}(x)) < \epsilon, \quad (10)$$

of width $m + p + 2$ and whose depth is recorded in Table 1.

We also substantially sharpened the variant of the above rates when the training and testing data belong to an f -normalizable and n -efficient dataset.

Corollary 24 (Hyperbolic Neural Networks are Efficient Universal Approximators)

Consider the setting of Corollary 23 and let $n \in \mathbb{N}_+$, σ be a non-affine piecewise linear activation function, and let \mathcal{X} be an f -normalizable and n -efficient dataset for f . Then, there is a $W \in \mathbb{N}_+$, a $g \in \mathcal{NN}_{p,m,W}^\sigma$, and a constant $\kappa > 0$ not depending on ϵ , p , or on m , such that the hyperbolic feedforward network: $\hat{f} \triangleq \text{Exp}_{\mathbb{D}_c^m, f(0)} \circ g \circ \text{Exp}_{\mathbb{D}_c^p, 0}^{-1}$, satisfies the approximation bound:

$$\sup_{x \in \mathcal{X}} d_c(f(x), \hat{f}(x)) < \kappa \sqrt{m\epsilon}.$$

Furthermore, the DNN g satisfies the sub-exponential complexity estimates:

- (i) **Width:** satisfies $m \leq W \leq m(4p + 10)$,
- (ii) **Depth:** of order $\mathcal{O} \left(m + m\epsilon^{\frac{2p}{3(np+1)} - \frac{p}{np+1}} \right)$,
- (iii) **Number of trainable parameters:** is of order $\mathcal{O} \left(m(m^2 - 1)\epsilon^{-\frac{2p}{3(np+1)}} \right)$.

3.4.2 UNIVERSAL SYMMETRIC POSITIVE-DEFINITE MATRIX-VALUED NETWORKS

Non-degenerate covariance matrices are fundamental tools for describing the non-trivial interdependence of various stochastic phenomena; with notable applications ranging from mathematical finance (Markowitz (1991)) to computer vision (see Haralick (1996)). Briefly, any covariance matrix A between p different random variables ξ_1, \dots, ξ_p can be identified (component-wise) with a $p \times p$ vector in the low-dimensional subset $P_p^+ \subset \mathbb{R}^{p \times p}$ given by:

$$P_p^+ \triangleq \{A \in \mathbb{R}^{p \times p} : (\forall x \in \mathbb{R}^p - \{0\}) x^\top A x > 0\};$$

here, we have identified $p \times p$ -matrices with vectors in p^2 via $(A_{i,j})_{i,j=1}^p \mapsto (A_{1,1}, \dots, A_{p,1}, \dots, A_{p,p})$. In fact, P_p^+ is a (non-linear) differentiable submanifold of $\mathbb{R}^{p \times p}$ (see Pennec et al. (2006)).

The Euclidean metric P_p^+ is not well-suited to the description of covariance matrices. For example, suppose that $\xi = (\xi_1, \dots, \xi_p)$ and $\zeta = (\zeta_1, \dots, \zeta_p)$ are vectors of features from some dataset of images. One would expect that, since the content of any image does not change if the image is rotated or shifted, then the relation between the covariance matrices $\text{Cov}(\xi)$ and $\text{Cov}(\zeta)$ should be equal to the distance of $\text{Cov}(X\xi + b) = X^\top \text{Cov}(\xi)X$ and $\text{Cov}(X\zeta + b) = X^\top \text{Cov}(\zeta)X$; where X is a $p \times p$ -orthogonal matrix and $b \in \mathbb{R}^p$ (since $x \mapsto Xx + b$ is exactly a rotation and shift in \mathbb{R}^p). However, this is not the case when comparing covariance matrices dissimilarity with the Euclidean distance.

In Pennec et al. (2006), a solution to this problem was obtained via the so-called “*affine-invariant*” metric. This distance function was obtained by equipping P_p^+ with a specific Cartan-Hadamard structure designed to encode invariances under the aforementioned symmetry. The distance function d_+ of this Riemannian metric satisfies $d_+(A, B) = d_+(X^\top AX, X^\top BX)$ for any $p \times p$ -orthogonal matrix X and any $A, B \in P_p^+$ and d_+ is computed via:

$$d_+(A, B) \triangleq \left\| \sqrt{A} \log \left(\sqrt{A}^{-1} B \sqrt{A}^{-1} \right) \sqrt{A} \right\|_2;$$

where $\|\cdot\|_2$ is the Fröbenius norm on $\mathbb{R}^{p \times p}$, $\log(\cdot)$ is the inverse of the matrix exponential $\exp(\cdot)$ and $\sqrt{\cdot}$ is the matrix square-root (both of which are well-defined on P_p^+). The Riemannian exponential maps is obtained as follows. Identify $\mathbb{R}^{p(p+1)/2}$ with the set of Sym_p of $p \times p$ -symmetric-matrices:

$$\text{Sym}_p : \mathbb{R}^{p(p+1)/2} \ni (a_{1,1}, \dots, a_{1,p}, \dots, a_{p,p}) \mapsto \begin{pmatrix} a_{1,1} & \dots & a_{1,p} \\ \vdots & \ddots & \vdots \\ a_{1,p} & \dots & a_{p,p} \end{pmatrix} \in \text{Sym}_p. \quad (11)$$

Under this identification, the Riemannian exponential map and its inverse are computed to be:

$$\begin{aligned} \text{Exp}_{P_p^+, A}(B) &= \sqrt{A} \exp \left(\sqrt{A}^{-1} \text{Sym}_p(B) \sqrt{A}^{-1} \right) \sqrt{A} \\ \text{Exp}_{P_p^+, A}(B)^{-1} &= \text{Sym}_p^{-1} \left[\sqrt{A} \log \left(\sqrt{A}^{-1} B \sqrt{A}^{-1} \right) \sqrt{A} \right]. \end{aligned} \quad (12)$$

The suitability of this geometry to the problem of covariance-matrix feature description is well-studied, especially in the computer vision literature. Most relevant to our program, in Meyer et al. (2011b) the authors introduce a class of non-Euclidean regression models on P_p^+ and in Bonnabel (2013) and Bécigneul and Ganea (2018) classes of optimization algorithms were introduced which leverage the Riemannian geometry of P_p^+ . Likewise, there have been numerous optimization software advances specifically designed to handle such situations Boumal et al. (2014), Townsend et al. (2016), and Miolane et al. (2020).

Subsequently, Baes et al. (2019) and Herrera et al. (2020) extended some of these ideas by introducing geometric deep learning models with inputs and outputs from the set of $p \times p$ -symmetric

positive *semi*-definite matrices to itself. Thereafter, in [Kratsios and Bilokopytov \(2020\)](#) the authors derived a universal extension of the regression model of [Meyer et al. \(2011b\)](#) which necessarily inputs and outputs matrices from P_p^+ and P_m^+ , respectively. The latter model class of “affine-invariant” GDNs have the representation:

$$\hat{f} \triangleq \text{Exp}_{P_m^+, I_m} \circ g \circ \text{Exp}_{P_p^+, I_p}^{-1} \in C(P_p^+, P_m^+); \quad (13)$$

where, $g \in \mathcal{NN}_{\frac{p(p+1)}{2}, \frac{m(m+1)}{2}}^\sigma$. The following are, respectively, quantitative and efficient improvements of the universal approximation theorems for the model class (13) derived in [\(Kratsios and Bilokopytov, 2020, Section 3.2.1\)](#). We denote the set of $p \times p$ -orthogonal matrices by $O(p)$.

Corollary 25 (Universality of the Affine-Invariant Networks of (13)) *Let $\sigma \in C(\mathbb{R})$ satisfy the Kidger-Lyons conditions. Fix $\epsilon > 0$, $f \in C(P_p^+, P_m^+)$, and a non-empty compact subset $K \subseteq P_m^+$. Then, there exists an “affine-invariant” GDN \hat{f} satisfying:*

$$\max_{A \in K} \sup_{X \in O(p)} d_+ \left(X^\top f(A)X, X^\top \hat{f}(A)X \right) < \epsilon. \quad (14)$$

Moreover, g in the representation (13), has width at-most $\frac{p(p+1)+m(m+1)+4}{2}$ and its depth is recorded in [Table 1](#) with $2^{-1}p(p-1)$ and $2^{-1}m(m-1)$ in place of p and m , respectively.

Using the concept of efficient dataset, we are able to refine the above theorem.

Corollary 26 (Affine-Invariant GDNs are Efficient Universal Approximators)

Consider the setting of [Corollary 25](#) and let $n \in \mathbb{N}_+$, σ be a non-affine piecewise linear activation function, and let \mathcal{X} be an f -normalizable and n -efficient dataset for f . Then, there is a $W \in \mathbb{N}_+$, a $g \in \mathcal{NN}_{2^{-1}p(p+1), 2^{-1}m(m+1); W}^\sigma$, and a constant $\kappa > 0$ not depending on ϵ , p , or on m , such that the GDN $\hat{f} \triangleq \text{Exp}_{P_m^+, f(0)}^{-1} \circ g \circ \text{Exp}_{P_p^+, 0}^{-1}$, satisfies the approximation bound:

$$\sup_{x \in K} \sup_{X \in O} d_+ \left(X^\top f(x)X, X^\top \hat{f}(x)X \right) < \frac{\kappa \sqrt{m(m+1)}}{\sqrt{2}} \epsilon.$$

Furthermore, g satisfies the sub-exponential complexity estimates:

- (i) **Width:** satisfies $m \leq W \leq 2^{-1}m(m+1)(2p(p+1)+10)$,
- (ii) **Depth:** of order $\mathcal{O} \left(m(m+1) \left(1 + \epsilon^{\frac{p(p+1)}{3(2^{-1}np(p+1)+1)} - \frac{p(p+1)}{np(p+1)+2}} \right) \right)$,
- (iii) **Number of trainable parameters:** of order $\mathcal{O} \left(m(m+1) \left((m(m+1))^2 - 1 \right) \epsilon^{-\frac{2p(p+1)}{3(np^2+np+2)}} \right)$.

3.4.3 SPHERICAL NEURAL NETWORKS AND APPROXIMATION IN KENDALL’S PRE-SHAPE SPACE

Our last illustration focuses on [Theorem 6](#). As described in [Straub et al. \(2015\)](#), spherical data plays a central role in many computer vision applications as a natural medium for describing direction data. This, and its connections to various other areas such as geo-statistics, has made learning from spherical data an active area of research both in the machine learning (see [Dutordoir et al. \(2020\)](#), and [Hamsici and Martinez \(2007\)](#)), and in the statistics communities (see [Dai and Müller \(2018a\)](#)).

Geodesics on the sphere are well-studied; for example, the distance on S^p is $d_{S^p}(x, y) = \arccos(y^\top x)$. Most importantly for our analysis, the Riemannian exponential map, and its inverse, at any $x \in S^p$ admits the following closed-form expressions

$$\text{Exp}_{S^p, x}(v) = \cos(\|v\|)x + \sin(\|v\|) \frac{v}{\|v\|} \quad \text{and} \quad \text{Exp}_{S^p, x}(y)^{-1} = \frac{y - (y^\top x)x}{\|y - (y^\top x)x\|} \arccos(y^\top x), \quad (15)$$

(see (Dai and Müller, 2018a, page 3341) for example). Unlike the geometries in the two previous examples, the sphere is positively curved, with sectional curvature always equal to 1. Consequentially the Riemannian Exponential map’s inverse, about any point $x \in S^p$, is not globally defined.

Corollary 27 (Local Quantitative Deep Universal Approximation for Spherical Data) *Let $\sigma \in C(\mathbb{R})$ satisfy Assumption 1. For any continuous function $f : S^p \rightarrow S^m$, any $\epsilon > 0$, given any $B_{S^p}(x, \delta) \subseteq S^p$ for which $0 < \delta < \pi$ then, for every $g \in \mathcal{NN}_{p,m,p+m+2}^\sigma$ the GDN $\hat{f} \triangleq \text{Exp}_{S^m, f(x)} \circ g \circ \text{Exp}_{S^p, x}^{-1}$, is well-defined on $\overline{B_{S^p}(x, \delta)}$ and there is one such \hat{f} satisfying the approximation bound:*

$$\max_{x \in \overline{B_{S^p}(x, \delta)}} d_{S^m} \left(f(x), \hat{f}(x) \right) \leq \epsilon.$$

Moreover, g ’s depth is recorded in Table 1.

Remark 28 *The quantity π estimating the maximum radius of the ball $B_{S^p}(x, \delta)$ in Corollary 27 is a lower-bound for $\mathcal{U}_f(x)$. This estimate is specific to the sphere’s geometry, where we have tightened the generic estimate of (4) via Klingenberg (1968)’s Quarter-Pinched Sphere Theorem.*

The obstruction identified in Theorem 6 has the following consequence for spherical spaces.

Corollary 29 (Universal Approximation on Spheres is Local) *If $\mathcal{X} = \mathcal{Y} = S^m$, then there exists a non-empty compact subset $\mathcal{K} \subseteq S^m$, $x \in \mathcal{K}$, $\epsilon > 0$, such that for every $k \in \mathbb{N}$, and for every $g \in \mathcal{NN}_{m,m;k}^\sigma$, the map $\text{Exp}_{S^m, f(x)} \circ g \circ \text{Exp}_{S^m, x}^{-1}$ is a well-defined function in $C(\mathcal{K}, \mathcal{Y})$, but*

$$\inf_{\substack{g \in \mathcal{NN}_{m,m,k}^\sigma \\ y \in S^m, k \in \mathbb{N}_+}} \sup_{z \in \mathcal{K}} d_{\mathcal{Y}} \left(\text{Exp}_{S^m, f(x)} \circ g \circ \text{Exp}_{S^m, x}^{-1}(z), 1_{S^m}(z) \right) \geq \epsilon.$$

Remark 30 (Universal Approximation Theorem in Kendall’s Pre-Shape Space) *As shown in Kendall (1984) and Le and Kendall (1993), high-dimensional spheres coincide with Kendall’s pre-shape space. Therefore, Corollary 29 guarantees that GDNs between Kendall’s pre-shape spaces are universal. These GDNs are a direct “deep learning” extension of the Procrustean (pre-shape space) regressors of Thomas Fletcher P. (2013).*

4. Main Results on Building Universal GDL Models using GDNs

Next, we treat GDNs as elementary building blocks, and we derive several results which describe how to combine GDNs to build universal *geometric deep learning models* compatible with complicated geometries; summarized in Figure 5. This additional flexibility is gained by combining multiple GDNs using “*geometric processing layers*” (symbolized arrows in Figure 5 other than \hat{f}_i). These are non-trainable layers that encode specific geometric “features” into our geometric deep learning models, such as products, quotients, parameterization, or boundary-like regions.

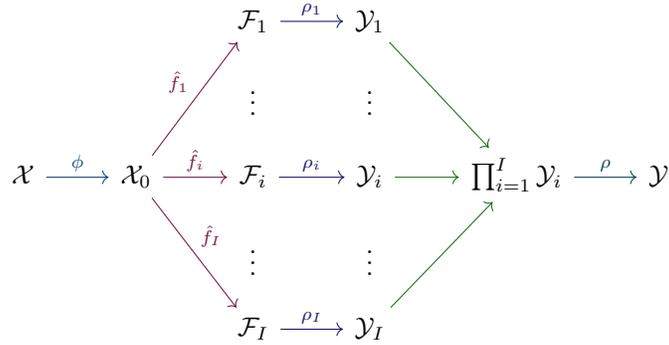


Figure 5: The geometric deep learning model’s full computational graph.

We briefly outline Figure 5: ϕ is a *feature map* with the UAP-invariance property of Kratsios and Hyndman (2021), each of the \hat{f}_i are GDNs mapping into “deep feature spaces \mathcal{F}_i ”, the ρ_i are “good” quotient maps which impose symmetries on the deep features in \mathcal{F}_i , the green arrows are a parallelization of the architectures thus far via a “skip connection” (analogously to He et al. (2016) and Srivastava et al. (2015)), and ρ parameterizes the output space \mathcal{Y} up to a “negligible subset of \mathcal{Y} ” (where our notion of negotiability is similar to that of Toruńczyk (1978) and to van Mill (2001)).

We progressively introduce each geometric processing layer in Figure 5 and incrementally derive its universal approximation theorem. Each step of our derivative will correspond to a geometric deep learning model defined by a computational sub-graph of Figure 5.

4.1 Feature Spaces and Quotient Layers: For Quotient Geometries

Often, a metric space \mathcal{Y} ’s geometry is extremely complicated, but its description can substantially be simplified by understanding its points as equivalence classes of symmetries defined on a “simpler” m -dimensional Riemannian manifold \mathcal{F} . We consider the situation of Figure 6.

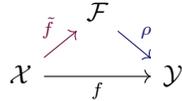


Figure 6: Factorizing the Target Function.

Our setting is formalized as follows. Let $G \subset C(\mathcal{F}, \mathcal{F})$ be a set of surjective *isometries*; that is, each $g \in G$ does not distort the relative distance between any two points $z_1, z_2 \in \mathcal{F}$ since $d_{\mathcal{F}}(z_1, z_2) = d_{\mathcal{F}}(g(z_1), g(z_2))$. We require the isometries in G to be “compatible” in the sense that:

Assumption 4 (Symmetric Space) $1_{\mathcal{F}} \in G$ and if $g_1, g_2 \in G$ then⁸ $g_1 \circ g_2^{-1} \in G$.

We consider output spaces \mathcal{Y} which are “invariant/symmetric to the isometries in G ”. Such \mathcal{Y} are called *symmetric spaces* and are widely studied both in the context of density estimation when data and/or parameters lie in a low-dimensional manifold in Li et al. (2020), non-linear dimension reduction Fletcher et al. (2004), learning faithful graph representations in Lopez et al. (2021),

8. We note that g_2^{-1} is always well-defined since every isometry is injective; thus, g_2 is a bijection and therefore it has a unique two-sided inverse g_2^{-1} .

stochastic filtering in [Pontier and Szpirglas \(1986\)](#), as well as several other instances in machine learning literature and its adjacent research areas.

Following ([Burago et al., 2001](#), Section 3.3), we do this by setting (resp. identifying) the points in \mathcal{Y} to be (resp. with) the equivalence classes: $[z] \triangleq \{z' : (\exists g \in G) g(z) = z'\}$. By ([Burago et al., 2001](#), Lemma 3.3.6), the set \mathcal{Y} is made into a *metric space* since the map:

$$\bar{d}_{\mathcal{F}}([z_1], [z_2]) = \inf_{g \in G} \bar{d}_{\mathcal{F}}(z_1, g(z_2)),$$

is a well-defined metric on \mathcal{Y} . Note that, $\bar{d}_{\mathcal{F}}([z_1], [z_2]) \leq d_{\mathcal{F}}(z_1, z_2)$ for any $z_i \in [z_i]$ and $i = 1, 2$. We call $\bar{d}_{\mathcal{F}}$ the *quotient metric* on \mathcal{Y} and \mathcal{Y} the *quotient metric space of \mathcal{F} generated by the symmetries in G* . If \mathcal{F} 's geometry is compatible with the symmetries described by G (Assumption 5 below), then the *projection map*:

$$\rho_1 : \mathcal{F} \ni z \mapsto [z] \in \mathcal{Y}, \quad (16)$$

implies that: “locally, \mathcal{F} looks like a disjoint union of identical pieces of \mathcal{Y} and that it looks the same everywhere”. Following ([Burago et al., 2001](#), Proposition 3.4.15.), this happens when:

Assumption 5 (Compatibility between G and \mathcal{F})

- (i) For each $z \in \mathcal{F}$ there is a $k_z > 0$ such that if $d(z, g(z)) < k_z$ then g is the identity,
- (ii) For each $z \in \mathcal{F}$ and each $g \in G$ if $g \neq 1_{\mathcal{F}}$ then $z \neq g(z)$.

Example 5 (([Hatcher, 2002](#), Exercise 1.3.23)) If G is finite, then Assumption 5 (ii) holds.

The deep feature space must be connected by paths and every such path is topologically comparable.

Assumption 6 (The Deep Feature Space is Simply Connected) The deep feature space \mathcal{F} is connected and simply connected.

Example 6 The Euclidean space \mathbb{R}^p , the space of symmetric positive definite matrices, and the hyperbolic space each satisfy Assumption 6. Nevertheless, if $p \geq 1$, then Examples 2 and 3 shows that S^p is still much more complicated than \mathbb{R}^p , topologically.

We arrive at the following quantitative non-Euclidean universal approximation theorem.

Theorem 31 (Universal Approximation for Quotient Metric Spaces) Suppose that Assumptions 4 and 5 hold, and that Assumption 6 also holds for \mathcal{X} and for \mathcal{F} and let $\sigma \in C(\mathbb{R})$ satisfy Assumption 1. Let $\epsilon > 0$, $x \in \mathcal{X}$, $f \in C(\mathcal{X}, \mathcal{Y})$, $\mathbb{X} \subset B_{\mathcal{X}}(x, \mathcal{U}_f(x))$ be a compact dataset, and $\rho_1 : \mathcal{F} \ni z \mapsto [z] \in \mathcal{Y}$. There is a GDN \hat{f} with representation: $\hat{f} = \text{Exp}_{\mathcal{F}, f} \circ g \circ \text{Exp}_{\mathcal{X}, x}^{-1}$, for some $y \in \mathcal{F}$, where $g \in \mathcal{NN}_{p, m, W}^{\sigma}$ such that:

$$\sup_{x \in \mathbb{X}} \bar{d}_{\mathcal{F}}(f(x), \rho_1 \circ \hat{f}(x)) < \epsilon. \quad (17)$$

Moreover, $W = p + m + 2$ and g 's depth is as in Table 1.

EXAMPLES: UNIVERSAL APPROXIMATORS WITH COMPUTATIONAL SUBGRAPH OF FIGURE 6

The following example is an essential component of the projective shape space introduced by [Mardia and Patrangenaru \(2005\)](#). We return to the following manifold in Section 4.4.

Example 7 (Universal Approximators to the Real Projective Space ($\mathbb{R}P^m$)) An element of the real projective space $\mathbb{R}P^m$ is a line in \mathbb{R}^{m+1} passing through the origin. Since every such line is determined by its intersections with $S^m \triangleq \{x \in \mathbb{R}^{m+1} : \|x\| = 1\}$ then elements of $\mathbb{R}P^m$ are:

$$[x] \triangleq \{\{x, -x\} : x \in S^m\};$$

where, $G = \{1_{S^m}, [x \mapsto -x]\}$. Furthermore, in (Hatcher, 2002, Example 1.43), it is shown that G and S^m satisfy Assumptions 4 and 5. By Example 6, \mathbb{R}^p and S^m satisfy Assumption 6. Thus, the projection map $\rho_1 : S^m \ni y \mapsto [y] \in \mathbb{R}P^m$ verified the conditions of Theorem 31 and $\mathbb{R}P^m$ is a quotient of S^m by the symmetry defined by G . Since Corollary 29 implies that deep neural models

$$\{\text{Exp}_{S^m, f(x)} \circ g : g \in \mathcal{NN}_{p,m}^\sigma\},$$

are locally universal in $C(\mathbb{R}^p, S^m)$, then Theorem 31 implies that each $f \in C(\mathbb{R}^p, \mathbb{R}P^m)$ can be locally approximated by a deep neural model of the form $\rho_1 \circ \text{Exp}_{S^m, f(x)} \circ g$.

Our next illustration of Theorem 31 concerns universal approximators into the “flat torus”. Examples of the torus geometry in data visualization in Li (2004) and in Maron et al. (2017).

Example 8 (Universal Approximators on the Flat Torus (\mathbb{T}^m)) Let G be the “integer lattice translations”: $G \triangleq \{g : \mathbb{R}^m \ni z \mapsto z + k \in \mathbb{R}^m\}_{k \in \mathbb{Z}^m}$. Then G satisfies Assumption 4 and \mathbb{R}^m satisfies Assumption 5. Example 6 states that \mathbb{R}^p and \mathbb{R}^m satisfy Assumption 6. The classes in \mathcal{Y} are therefore in correspondence with points in the cube $[0, 1]^m$ but the distance between any $y_1, y_2 \in \mathcal{Y}$ is the “flat toral distance”:

$$\bar{d}_{\mathbb{R}^m}(y_1, y_2) = \inf_{z \in \mathbb{Z}^m} \|y_1 - (y_2 + z)\|.$$

In this space, we are allowed to “teleport” along nodes in integer lattice \mathbb{Z}^m but every other movement counts. It is a standard exercise to show that the above G satisfies Assumptions 4 and Assumption 5. Thus, Theorem 31 implies that for every $f \in C(\mathbb{R}^p, \mathbb{T}^m)$ there is a DNN $\hat{f} \in \mathcal{NN}_{p,m}^\sigma$ such that $\rho_1 \circ \hat{f}$ locally approximates f .

4.2 Skip Connections and Parallelization: For Product Geometries

In Gribonval Rémi et al. (2021), the authors describe a calculus for “parallelizing” several feedforward networks $g_1, \dots, g_I \in C(\mathbb{R}^p, \mathbb{R})$ to efficiently form a deep neural model in $C(\mathbb{R}^p, \mathbb{R}^I)$. There, the parallelized model implements the map:

$$\mathbb{R}^p \ni x \mapsto (g_1(x), \dots, g_I(x)) \in \mathbb{R}^I. \quad (18)$$

In Cheridito et al. (2021), the author gave conditions on the activation function σ under which the map (18) could be implemented by a single feedforward network. Otherwise, the map of (18) are I different learning models defined by a more complicated computational graph where the last layer can be understood as a sort of “skip connection”. Note that, most commonly used deep learning software such as, Abadi et al. (2015) and Team et al. (2016), are designed to handle these types of computational graphs.

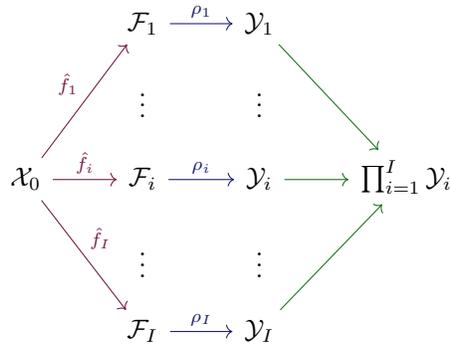


Figure 7: Parallelized Computational Graph.

In the geometric deep learning situation, *parallelization* is even more interesting since it allows us to simultaneously generated predictions on potentially very different output spaces. Building on the ideas of Section 4.1, let $\mathcal{Y}_1, \dots, \mathcal{Y}_I$ be metric spaces and suppose that there are deep feature spaces $\mathcal{F}_1, \dots, \mathcal{F}_I$ such that each satisfies Assumptions 4 and 5 (with \mathcal{F} and \mathcal{Y} respectively replaced by \mathcal{F}_i and \mathcal{Y}_i). Extending (18), we consider the problem of approximating functions in $C(\mathcal{X}, \prod_{i=1}^I \mathcal{Y}_i)$ where the *product* is defined by $\prod_{i=1}^I \mathcal{Y}_i \triangleq \{(y_1, \dots, y_I) : y_i \in \mathcal{Y}_i\}$. As usual, we equip $\prod_{i=1}^I \mathcal{Y}_i$ with the *product-metric* defined for $(y_1, \dots, y_I), (\tilde{y}_1, \dots, \tilde{y}_I) \in \prod_{i=1}^I \mathcal{Y}_i$ by:

$$d_{\prod_i \mathcal{Y}_i}((y_1, \dots, y_I), (\tilde{y}_1, \dots, \tilde{y}_I)) \triangleq \max_{i=1, \dots, I} \{d_{\mathcal{Y}_i}(y_i, \tilde{y}_i)\};$$

where $d_{\mathcal{Y}_i}$ denotes the metric on \mathcal{Y}_i for $i = 1, \dots, I$.

Remark 32 (Notation) *The dimension of each \mathcal{F}_i is denoted by m_i and $f_i \in C(\mathcal{X}, \mathcal{Y}_i)$ denotes the composition $f \in C(\mathcal{X}, \mathcal{Y})$ and the canonical projection $\mathcal{Y} \ni (y_1, \dots, y_I) \mapsto y_i \in \mathcal{Y}_i$. We also use $\rho_k : \mathcal{F}_k \ni z \mapsto [z] \in \mathcal{Y}_k$ to denote the projection maps discussed in the previous section.*

Corollary 33 (Universality of Parallelized GDNs) *Suppose that \mathcal{Y}_i and \mathcal{F}_i satisfy Assumptions 4, 5, and 6 (mutatis mutandis). Let $\sigma \in C(\mathbb{R})$ satisfy Assumption 1, $f \in C(\mathcal{X}, \prod_{i=1}^I \mathcal{Y}_i)$, and fix $\epsilon > 0$. Fix: $0 < \delta < \min_{i=1, \dots, I} \{\mathcal{U}_{f_i}(x)\}$. Then, for each $x \in \mathcal{X}$ and each compact dataset $\mathcal{X} \subseteq B_{\mathcal{X}}(x, \delta)$ there exist $g_i \in \mathcal{NN}_{p, m_i}^\sigma$ (for $i = 1, \dots, I$) such that:*

$$\hat{f}_i \triangleq \text{Exp}_{\mathcal{Y}_i, f_i(x)} \circ g_i \circ \text{Exp}_{\mathcal{X}, x}^{-1}, \quad (19)$$

satisfy the estimate:

$$\sup_{x \in \mathcal{X}} d_{\prod_i \mathcal{Y}_i} \left(f(x), (\rho_1 \circ \hat{f}_1(x), \dots, \rho_I \circ \hat{f}_I(x)) \right) < \epsilon. \quad (20)$$

Moreover, the complexity of each \hat{f}_i depends on \mathcal{X} 's geometry as follows:

(i) **Efficient Case:** *If there is an $n \in \mathbb{N}_+$ such that \mathcal{X} is f_i -normalized, n -efficient for f_i , $\rho_i = 1_{\mathcal{F}_i}$, and if σ is piecewise linear then:*

- (i) **Width:** *satisfies $m_i \leq W \leq m(4p + 10)$,*
- (ii) **Depth:** *of order $\mathcal{O} \left(m_i + m_i \epsilon^{\frac{2p}{3(np+1)} - \frac{p}{np+1}} \right)$,*
- (iii) **Number of trainable parameters:** *is of order $\mathcal{O} \left(m_i(m_i^2 - 1) \epsilon^{-\frac{2p}{3(np+1)}} \right)$.*

Moreover, in this setting, the right-hand side of (20) is instead $\kappa \epsilon \sum_{i=1}^I \sqrt{m_i}$; where $\kappa > 0$ is a constant not depending on p , m_i , or on ϵ .

(ii) **General Case:** *If \mathcal{X} is not efficient for f_i , $\rho_i \neq 1_{\mathcal{F}_i}$, or \mathcal{X} is not f_i -normalized, then $W = p + m_i + 2$ and each g_i has depth as in Table 1 (but with m_i in place of m).*

Remark 34 *In Corollary 33 (i), $\rho_i = 1_{\mathcal{F}_i}$ implies that $\mathcal{F}_i = \mathcal{Y}_i$ for each $i = 1, \dots, I$.*

4.3 UAP-Preserving Layers: For Embedded Geometries and Parameterization

We require that the ‘‘feature map’’ ϕ has the UAP-invariance property, which means that pre-composing the learning model by ϕ does not negatively impact the learning model’s universal approximation property (UAP). The following condition is sufficient and the condition is known to be sharp in a broad range of cases (see Kratsios and Bilokopytov (2020)).

Assumption 7 (UAP-Invariant Feature Map) $\phi : \mathcal{X} \rightarrow \mathcal{X}_0$ is continuous and injective.

Example 9 (UAP-Invariant Feature Maps When \mathcal{X} is Embedded in \mathbb{R}^p) If \mathcal{X} is compact and embedded in \mathbb{R}^p , then the reconfiguration networks of [Kratsios and Hyndman \(2021\)](#) and the injective ReLU networks of [Puthawala et al. \(2020\)](#) are both classes of UAP-invariant feature maps.

Dually, UAP-Invariant Readout maps allow us to extend any universal approximation result to any space which is “almost parameterized by ρ ”. Assumption 8 below, is the dual form of the UAP-invariant feature condition, above. It both extends and significantly simplifies the condition of ([Kratsios and Bilokopytov, 2020](#), Assumption 3.2). The key point is to reinterpret [Toruńczyk \(1978\)](#)’s “homotopy negligible sets” and the \mathcal{Z} -sets of ([van Mill, 2001](#), Section 5).

Assumption 8 (UAP-Invariant Readout Map) A map $\rho : \prod_{i=1}^I \mathcal{Y}_i \rightarrow \mathcal{Y}$ is said to be a UAP-invariant readout map if:

- (i) ρ is continuous and admits a continuous right-inverse R on its image $\text{Im}(\rho) \subseteq \mathcal{Y}$,
- (ii) There is a (homotopy) $H \in C([0, 1] \times \mathcal{Y}, \mathcal{Y})$ satisfying:
 - (a) For each $0 \leq t < 1$ and every $y \in \mathcal{Y}$ we have: $H_t(y) \in \text{Im}(\rho)$,
 - (b) For every $\epsilon > 0$, there is a $t_\epsilon \in [0, 1)$ satisfying:

$$\sup_{y \in \mathcal{Y}} d_{\mathcal{Y}}(H_{t_\epsilon}(y), y) < \epsilon.$$

The simplest non-trivial instance of an interesting class of UAP-invariant readout maps arises from projecting the output of a GDL model taking values in a Euclidean space onto a non-empty closed and convex subset thereof. Furthermore, this class trivially satisfies Assumption 8 (ii).

Example 10 (Projections onto Closed Convex Sets are UAP-Invariant Readout Maps)

Let $\mathcal{Y} \subseteq \mathbb{R}^p$ be non-empty, closed, and convex. By ([Bauschke and Combettes, 2017](#), Theorem 3.16) the metric projection $P_{\mathcal{Y}}$ onto \mathcal{Y} defined by:

$$P_{\mathcal{Y}}(x) := \underset{\tilde{y} \in \mathcal{Y}}{\text{argmin}} \|y - \tilde{y}\|,$$

is a well-defined, 1-Lipschitz surjection of \mathbb{R}^p onto \mathcal{Y} . A direct computation confirms that the inclusion map $R : \mathcal{Y} \ni z \mapsto z \in \mathbb{R}^p$ is a continuous right-inverse for $P_{\mathcal{Y}}$; thus, Assumption 8 (i) is satisfied. Since $\mathcal{Y} - \text{Im}(P_{\mathcal{Y}}) = \emptyset$ then, we may take $H_t(y) := y$ to be the homotopy in Assumption 8 (ii). Hence, the metric projection $P_{\mathcal{Y}}$ is a UAP-invariant readout map if \mathcal{Y} is a non-empty, closed and convex set.

In Example 10, the right-inverse of the projection map $\pi_{\mathcal{Y}}$ is never a continuous two-sided inverse, i.e. is a homeomorphism, if \mathcal{Y} is bounded⁹. In particular, it never has a smooth two-sided inverse on its image, which can of-course be advantageous while training.

Nevertheless, it can be preferable to instead map \mathbb{R}^p homeomorphically onto \mathcal{Y} ’s interior, provided that \mathcal{Y} has non-empty interior, and simply disregard \mathcal{Y} ’s boundary. It turns out that this is possible by appealing to \mathcal{Y} ’s gauge, also called \mathcal{Y} ’s Minkowski functional, as is outlined by the next example.

Example 11 (UAP-Invariant Readouts on Convex Bodies via Gauges) Let \mathcal{Y} be a convex subset of \mathbb{R}^p with non-empty interior containing 0. Following [Kriegel and Michor \(1997\)](#), a gauge $\mu_{\mathcal{Y}}$ of a bounded convex set \mathcal{Y} centered containing 0 is the real-valued function defined on $x \in \mathbb{R}^p$ by:

$$\mu_{\mathcal{Y}}(x) \triangleq \inf\{\lambda > 0 : x \in \lambda\mathcal{Y}\}.$$

9. Since this would lead to a contraction of the non-compactness of \mathbb{R}^p .

Using \mathcal{Y} 's gauge, we may define the map:

$$\rho : \mathbb{R}^p \ni y \mapsto \frac{1}{1 + \mu_{\mathcal{Y}}(y)} y \in \text{int}(\mathcal{Y}); \quad (21)$$

which is in fact a continuous bijection with continuous inverse given by $z \mapsto \frac{1}{1 - \mu_{\mathcal{Y}}(z)} z$. In other words, ρ is a homeomorphism between \mathbb{R}^p and $\text{int}(\mathcal{Y})$; in particular, Assumption 8 (i) holds.

It remains to show that \mathcal{Y} 's boundary is negligible, in the sense of Assumption 8 (ii). For this we observe that, the convexity of \mathcal{Y} and the fact that $0 \in \text{int} \mathcal{Y}$ implies that each $y \in \mathcal{Y}$ is identified with the unique line segment $\gamma_{[0,y]} : [0, 1] \rightarrow \mathcal{Y}$ satisfying: $\gamma_{[0,y]}(0) = 0$, $\gamma_{[0,y]}(1) = y$, and such that $\gamma_{[0,y]}(t) \in \text{int} \mathcal{Y}$ whenever $0 \leq t < 1$. Therefore, the following homotopy “pushing \mathcal{Y} towards 0”:

$$H_t(y) \triangleq ty; \quad (22)$$

satisfies Assumption 8 (ii).

NB, a benefit of the readout map ρ defined in (21) over the readout map defined in Example 10 is that ρ and its two-sided inverse are often differentiable on most of \mathbb{R}^p ; which is of course convenient for training. More precisely, by the implicit function theorem, ρ and ρ^{-1} are continuously differentiable on $\mathbb{R}^p - \{0\}$ if and only if μ is. Since \mathcal{Y} is convex then, μ defines a norm on \mathbb{R}^p by (Narici and Beckenstein, 2011, Exercise 5.105) and therefore by (Kriegel and Michor, 1997, Proposition 13.14) μ is k -times continuously differentiable on $\mathbb{R}^p - \{0\}$ if and only if \mathcal{Y} has a C^k -boundary.

Besides illustrating the non-vacuousness of Assumption 8 (ii), Example 11 suggests that a UAP-invariant readout map's must be “topologically generic”¹⁰ and surjective thereon up to \mathcal{Y} 's boundary. This is indeed the case whenever \mathcal{Y} is a Riemannian manifold with boundary, as implied by the following geometric description of UAP-invariant readout maps' images.

Proposition 35 (Geometric Description of UAP-Invariant Readout Map's Images)

Suppose that ρ satisfies Assumption (8). Then:

- (i) ρ 's Image is Topologically Generic in \mathcal{Y} : $\text{Im}(\rho)$ is a dense open subset of \mathcal{Y} ,
- (ii) ρ 's Remainder Belongs to \mathcal{Y} 's Boundary: If \mathcal{Y} is a topological manifold whose topology is induced by the metric $d_{\mathcal{Y}}$ then, $\mathcal{Y} - \text{Im}(\rho)$ is contained in \mathcal{Y} 's boundary.

Proposition 35 can be used to rule out maps ρ which are not UAP-invariant. In particular, we deduce the following necessary condition for UAP-invariant maps between Euclidean spaces.

Example 12 (UAP-Invariant Maps Between Euclidean Spaces are Surjective) Since \mathbb{R}^p is a topological manifold without boundary then, any $\rho : \mathbb{R}^p \rightarrow \mathbb{R}^p$ which is UAP-invariant must be surjective ρ since $\mathbb{R}^p - \text{Im}(\rho)$ must be contained in the empty set by Proposition 35 (ii).

We bring these concepts together in our final example of a UAP-invariant readout map, namely the softmax function of Bridle (1990) which is omnipresent in classification.

Example 13 (Softmax Function and the Simplex) Fix $C \in \mathbb{N}_+$ with $C \geq 2$, consider the closed convex set $\Delta_C \triangleq \{y \in [0, 1]^C : \sum_{c=1}^C y_c = 1\}$, and consider the Softmax function:

$$\text{Softmax}_C : \mathbb{R}^C \ni y \mapsto \left(\frac{e^{y_c}}{\sum_{c=1}^C e^{y_c}} \right)_{c=1}^C \in \text{int}(\Delta_C).$$

Define the affine map $W : \mathbb{R}^{C-1} \ni x \mapsto (x_1, \dots, x_{C-1}, 1) \in \mathbb{R}^C$ and define the map:

$$\rho : \mathbb{R}^{C-1} \ni y \mapsto \text{Softmax}_C \circ W(y) \in \text{int}(\Delta_C).$$

10. A “topologically generic” set here is meant in the sense of Baire Category; i.e. a dense G_δ -subset of \mathcal{Y} .

Then, $\rho : \mathbb{R}^{C-1} \rightarrow \text{int}(\Delta_C)$ is continuous, 1-Lipschitz, and a simple calculation verifies that $R(y) \triangleq (\ln(y_c) - \ln(y_C) + 1)_{c=1}^{C-1}$ is a continuous right-inverse of ρ defined on $\text{int}(\Delta_C)$. Thus, Assumption 8 (i) holds. Since Δ_C is convex and since ρ maps \mathbb{R}^{C-1} surjectively onto star-shaped set $\text{int}(\Delta_C)$ then, the following homotopy verifies Assumption 8 (ii)

$$H_t(y) := t(y - \bar{\Delta}) + \bar{\Delta},$$

where $\bar{\Delta} := (1/C, \dots, 1/C)$. Thus, ρ is a UAP-invariant readout map.

Our last result's statement is substantially simplified by considering *continuous*, but possibly sub-optimal, moduli of continuity. The relevant moduli of continuity are the following.

Remark 36 (Technical Notation regarding the Last Theorem) *In this case, for a uniformly continuous function $f : \mathcal{X} \rightarrow \mathcal{Y}$, between metric spaces \mathcal{X} and \mathcal{Y} , with (possibly discontinuous) modulus of continuity $\omega(f, \cdot)$ we define a continuous modulus of continuity: $\tilde{\omega}_f$ as follows. If the modulus of continuity $\omega(f, \cdot)$ of f is continuous on $[0, \infty)$ then set $\tilde{\omega}_f \triangleq \omega(f, \cdot)$ otherwise, set $\tilde{\omega}_f \triangleq \lim_{\tilde{t} \downarrow t} \tilde{t}^{-1} \int_{\tilde{t}}^{2\tilde{t}} \omega(f, s) ds$. We maintain this notation throughout the remainder of the paper.*

Example 14 *If f is Lipschitz, or more generally, Hölder then $\omega(f, \cdot) = \tilde{\omega}_f$.*

4.3.1 CONTROLLED UNIVERSAL APPROXIMATION: GENERAL VERSION

We may now state our final and main universal approximation theorem of this paper.

Theorem 37 (Controlled Universal Approximation: General Version) *Suppose that \mathcal{Y}_i and \mathcal{F}_i satisfy Assumptions 4, 5, and 6 (mutatis mutandis). Let $\sigma \in C(\mathbb{R})$ satisfy Assumption 1. Suppose also that ϕ and ρ are UAP-invariant. Fix $f \in C(\mathcal{X}, \mathcal{Y})$, $0 < \epsilon < 2^{-1} \sup_{t \in [0, \infty)} \omega_\rho(t)$, and fix a compact $\mathcal{X} \subseteq \mathcal{X}$ satisfying the following condition. There is an $x^* \in \mathcal{X}$ such that:*

$$\mathcal{X} \subset B_{\mathcal{X}}(x^*, \tilde{\omega}_\phi^{-1}(\eta \min_{i=1, \dots, I} \mathcal{U}_{[R \circ H_t \frac{\epsilon}{2} \circ f \circ \phi^{-1}]_i}(x^*))).$$

Then, for $i = 1, \dots, I$, there exist $g_i \in \mathcal{NN}_{p, m_i}^\sigma$ and $y_i \in \mathcal{Y}_i$ such that the model:

$$\hat{f} \triangleq \rho \left(\rho_1 \circ \text{Exp}_{\mathcal{F}_1, y_1} \circ g_1 \circ \text{Exp}_{\mathcal{X}, x}^{-1} \circ \phi, \dots, \rho_I \circ \text{Exp}_{\mathcal{F}_I, y_I} \circ g_I \circ \text{Exp}_{\mathcal{X}, x}^{-1} \circ \phi \right),$$

(whose computational graph is in Figure 5) satisfies the estimate:

$$\sup_{x \in \mathcal{X}} d_{\mathcal{Y}} \left(f(x), \hat{f}(x) \right) < \epsilon. \quad (23)$$

Moreover, the complexity of each g_i depends on \mathcal{X} 's geometry as follows:

- (i) **Efficient Case:** If there is an $n \in \mathbb{N}_+$ such that $\phi(\mathcal{X})$ is $f_i \circ \phi^{-1}$ -normalized, n -efficient for f_i , and if $\rho_i = 1_{\mathcal{F}_i}$ then each \hat{g}_i :
 - (a) **Width:** satisfies $m \leq W \leq m_i(4p + 10)$,
 - (b) **Depth:** of order $\mathcal{O} \left(m_i + m_i (\tilde{\omega}_\rho^{-1}(\frac{\epsilon}{2}))^{\frac{2p}{3(np+1)} - \frac{p}{np+1}} \right)$,
 - (c) **Number of trainable parameters:** is of order $\mathcal{O} \left(m_i(m_i^2 - 1) (\tilde{\omega}_\rho^{-1}(\frac{\epsilon}{2}))^{-\frac{2p}{3(np+1)}} \right)$,
 - (d) The right-hand side of (23) is instead $\kappa \epsilon \sum_{i=1}^I \sqrt{m_i}$; where $\kappa > 0$ is a constant not depending on p , m_i , or on ϵ .
- (ii) **General Case:** If \mathcal{X} is not efficient for f_i , $\rho_i \neq 1_{\mathcal{F}_i}$, or \mathcal{X} is not f_i -normalized, then $W = p + m_i + 2$ and each g_i has depth as in Table 2.

Regularity of σ	Order of Depth
$C^\infty(\mathbb{R}) + \text{Non-polynomial}$	$O\left(\frac{m_i(2 \operatorname{diam}(\phi(\mathbb{X})))^{2p}}{\kappa_2^{2p}(\omega^{-1}([R \circ H_{t_{2^{-1}\epsilon}} \circ f \circ \phi^{-1}]_i, \frac{\tilde{\omega}_\rho^{-1}(2^{-1}\epsilon)\kappa_1}{(1+\frac{p}{4})m_i}))^{2p}}\right)$
Non-affine polynomial ¹¹	$O\left(\frac{m_i(m_i+p)(2 \operatorname{diam}(\phi(\mathbb{X})))^{4p+2}}{\kappa_2^{4p+2}(\omega^{-1}([R \circ H_{t_{2^{-1}\epsilon}} \circ f \circ \phi^{-1}]_i, \frac{\tilde{\omega}_\rho^{-1}(2^{-1}\epsilon)\kappa_1}{(1+\frac{p}{4})m_i}))^{4p+2}}\right)$
$C(\mathbb{R}) + \text{Non-polynomial}$	$O\left(\frac{(\kappa_2\omega^{-1}([R \circ H_{t_{2^{-1}\epsilon}} \circ f \circ \phi^{-1}]_i, \frac{\tilde{\omega}_\rho^{-1}(2^{-1}\epsilon)\kappa_1}{2m_i(1+\frac{p}{4})}))^{-2p} m_i(2 \operatorname{diam}(\phi(\mathbb{X})))^{2p}}{\left(\kappa_2\omega^{-1}\left(\sigma, \frac{\tilde{\omega}_\rho^{-1}(2^{-1}\epsilon)}{(2 \operatorname{diam}(\phi(\mathbb{X})))^2[\omega^{-1}([R \circ H_{t_{2^{-1}\epsilon}} \circ f \circ \phi^{-1}]_i, \frac{\tilde{\omega}_\rho^{-1}(2^{-1}\epsilon)\kappa_1}{2m_i(1+\frac{p}{4})})]^{-2+1}}\right)}\right)^{2Bm_i(2)}}$

Where $\kappa_1, \kappa_2 >$ are independent of ϵ, p , and of m_i .

Table 2: Approximation Rates for Geometric Deep Learning Model in Figure 5

4.4 Applications

We use Theorem 37 to directly derive the UAP of various commonly implemented learning models.

4.4.1 DEEP SOFTMAX CLASSIFIERS ARE UNIVERSAL

Multiclass classification is one of the most common uses of deep learning. Here, the aim is to learn a function mapping \mathcal{X} to Δ_C , where C is the number of classes. The outputs of this function are typically interpreted as the probability that any input $x \in \mathcal{X}$ belongs to one of the C classes. Since most decision problems ultimately require the user to make a concrete decision as to which class(es) any $x \in \mathcal{X}$ belongs to. Thus, the most important outputs of any classifier are the 1-hot vectors (i.e., $y \in \Delta_C$ with 1 in a single coordinate and 0 elsewhere).

This problem is typically solved computationally by applying a softmax layer (see Example 13) to the output of a feedforward network $g \in \mathcal{NN}_{p,C}^\sigma$. The composite model $\rho \circ g$ is then trained to approximate the target classifier $c \in C(\mathbb{R}^p, \Delta_C)$.

We remark that it is clear that deep feedforward networks with softmax output layer can approximate any classifier taking values in the *interior* of the C -simplex; i.e. in:

$$\{y \in (0, 1)^C : \sum_{c=1}^C y_c = 1\}. \quad (24)$$

However, every 1-hot vector in Δ_C never belongs to (24) as it is in the boundary of the C -simplex. This topological obstruction has prevented uniform approximation results for continuous multiclass classifiers from appearing in the literature thus far. Nevertheless, Theorem 37 implies the result.

Corollary 38 (Deep Classifiers are Universal) *Let \mathbb{X} be a subset of a compact metric space \mathcal{X} , $C \in \mathbb{N}_+$, and σ satisfy Condition 1 and suppose that there exists a UAP-preserving feature map $\phi : \mathcal{X} \rightarrow \mathbb{R}^p$. For every $\epsilon > 0$ and every classifier f in $C(\mathcal{X}, \Delta_C)$ there is a $g \in \mathcal{NN}_{p,C}^\sigma$ satisfying:*

$$\max_{x \in \mathbb{X}} \sqrt{\sum_{c=1}^C \left(\frac{e^{g_c(x)}}{\sum_{\tilde{c}=1}^C e^{g_{\tilde{c}}(x)}} - f(x)_c \right)^2} < \epsilon.$$

Moreover the following complexity estimates hold, depending on \mathbb{X} and f :

- (i) **Efficient Case:** If \mathbb{X} is n -efficient for f , for some $n \in \mathbb{N}_+$ and f -normalized, then g is as in Theorem 20 (but with $2^{-1}\epsilon$ in place of ϵ),

- (ii) **General Case:** If \mathcal{X} is not efficient for f , then g has width $p + C + 1$ and depth recorded in Table 1 (but with $2^{-1}\epsilon$ in place of ϵ).

4.4.2 UNIVERSALITY OF THE DEEP KALMAN FILTER'S UPDATE RULE

Our next application concerns the approximation of unknown functions with *non-degenerate Gaussian measures*. We study a mild extension of the architecture implemented in the *deep Kalman filter* of [Krishnan et al. \(2015\)](#). Our analysis begins by first constructing a universal deep neural model which processes non-degenerate Gaussian measures. The set of non-degenerate Gaussian measures on \mathbb{R}^n , denoted by \mathcal{G}_n , consists of all Borel probability measures $\nu_{\mu,\Sigma}$ on \mathbb{R}^n with density

$$(2\pi)^{-\frac{n}{2}} \det(\Sigma)^{-\frac{1}{2}} e^{-\frac{1}{2}(\mathbf{x}-\mu)^\top \Sigma^{-1}(\mathbf{x}-\mu)};$$

where $\mu \in \mathbb{R}^n$ and Σ is a symmetric positive-definite $n \times n$ -matrix; the set of which is denoted P_n^+ .

There are various geometries on \mathcal{G}_n designed to highlight its different statistical properties while circumventing its non-linear structure. Notable examples include the restriction of the Wasserstein-2 distance from optimal-transport theory, see [Figalli \(2010\)](#), the Fisher-Rao metric introduced [Radhakrishna Rao \(1945\)](#) from information-geometry, and the invariant metric introduced in [Lovrić et al. \(2000\)](#) based on Lie-theoretic methods. We focus on the former due to its uses in modern adversarial learning, such as in [Arjovsky et al. \(2017\)](#); [Gulrajani et al. \(2017\)](#).

The Wasserstein-2 distances on the spaces of probability measures with finite-variance are notoriously challenging. However, [Dowson and Landau \(1982\)](#) found that when this distance is restricted to \mathcal{G}_n then it reduces to

$$\mathcal{W}_2(\nu_{\mu_1, \Sigma_1}, \nu_{\mu_2, \Sigma_2}) = \sqrt{\|\mu_1 - \mu_2\|^2 + \text{tr}\left(\Sigma_1 + \Sigma_2 - 2(\Sigma_1 \Sigma_2)^{\frac{1}{2}}\right)};$$

where $(\cdot)^{\frac{1}{2}}$ denotes the matrix-square-root. Following [Malagò et al. \(2018\)](#), the map ϕ_0 sending $\nu_{\mu,\Sigma} \in \mathcal{G}_n$ to $(\mu, \Sigma) \in \mathbb{R}^n \times P_n^+$ is not only a bijection, but it is also a homeomorphism when P_n^+ is equipped with the Fröbnius metric $d_F(A, B) \triangleq \sqrt{\sum_{i,j=1}^n (A_{i,j} - B_{i,j})^2}$. Building on the discussion of Section 3.4.2, we note that the map $(1_{\mathbb{R}^n} \times (\text{Sym}_n^{-1} \circ \log)) \circ \phi_0$, is a homeomorphism from \mathcal{G}_n to $\mathbb{R}^{n(n+1)/2}$ with inverse function given by:

$$\phi_{\mathcal{G}_n} \triangleq \phi_0^{-1} \circ (1_{\mathbb{R}^n} \times (\exp \circ \text{Sym}_n)); \quad (25)$$

where Sym_n parameterizes the set of symmetric $n \times n$ -matrices using $\mathbb{R}^{\frac{n(n+1)}{2}}$ and is defined in (11).

Corollary 39 (Universal Approximation with Gaussian Inputs/Outputs) *Let σ satisfy Assumption 1, $f \in C(\mathcal{G}_n, \mathcal{G}_m)$, fix an $\epsilon > 0$, and let $\mathcal{X} \subseteq \mathcal{G}_n$ be non-empty and compact. Then, there is a $g \in \mathcal{NN}_{n(n+1)/2, m(m+1)/2, (n(n+1)+m(m+1))/2+2}^\sigma$ satisfying:*

$$\sup_{\nu \in \mathcal{X}} \mathcal{W}_2(f(\nu), \phi_{\mathcal{G}_m}^{-1} \circ g \circ \phi_{\mathcal{G}_n}(\nu)) \leq \epsilon. \quad (26)$$

Moreover the following complexity estimates hold, depending on \mathcal{X} and f :

- (i) **Efficient Case:** If \mathcal{X} is n -efficient for f , for some $n \in \mathbb{N}_+$ and f -normalized, then g is as in Theorem 20 (but with $2^{-1}n(n+1)$ and $2^{-1}m(m+1)$ in place of p and m , respectively) and the right-hand side of (26) is $\kappa\sqrt{m(m+1)}\epsilon$, for a $\kappa > 0$ independent of m , n , and of ϵ .
- (ii) **General Case:** If \mathcal{X} is not efficient for f , then g has width $2^{-1}(n(n+1) + m(m+1)) + 2$ and depth recorded in Table 1 (but with $2^{-1}n(n+1)$ and $2^{-1}m(m+1)$ in place of p and m , respectively).

The architecture of Corollary 39 is an “uncontrolled version” of the architecture implementing the deep Kalman filter’s update rule. Fix $a, p \in \mathbb{N}$, at every increment, the deep Kalman filter of Krishnan et al. (2015) maps an observation $x_t \in \mathbb{R}^p$, an action $u_t \in \mathbb{R}^a$, and the previous latent state, which is a measure $\nu_t \in \mathcal{G}_n$, to a measure $\nu_{t+1} \in \mathcal{G}_n$ via a deep neural model of the form:

$$\nu_t = \hat{f}(x_{t-1}, u_{t-1}, \nu_{t-1}) \quad (\forall t \in \mathbb{N}_+), \quad (27)$$

where $\nu_0 \in \mathcal{G}_n$, $x_0 \in \mathbb{R}^p$, and $u_0 \in \mathbb{R}^a$ are fixed and \hat{f} is the deep neural model with representation:

$$\hat{f} = \phi_{\mathcal{G}_n}^{-1} \circ g \circ (1_{\mathbb{R}^p} \times 1_{\mathbb{R}^a} \times \phi_{\mathcal{G}_n}), \quad (28)$$

where $g \in \mathcal{NN}_{p+a+2^{-1}n(n+1), 2^{-1}n(n+1)}^\sigma$ and $\phi_{\mathcal{G}_n}$ is as in (25). Then a mild modification to Corollary 39 implies the universality of the “update map” of (28) defining the deep Kalman filter.

Corollary 40 (Universality of the Deep Kalman Filter’s Update Map) *Let σ satisfy Assumption 1, $f \in C(\mathbb{R}^p \times \mathbb{R}^a \times \mathcal{G}_n, \mathcal{G}_n)$, fix an $\epsilon > 0$, and let $\mathcal{X} \subseteq \mathbb{R}^p \times \mathbb{R}^a \times \mathcal{G}_n$ be non-empty and compact. Then, there is an $\hat{f} \in C(\mathbb{R}^p \times \mathbb{R}^a \times \mathcal{G}_n, \mathcal{G}_n)$ with representation (28) satisfying:*

$$\sup_{(x,u,\nu) \in \mathcal{X}} \mathcal{W}_2(f(x,u,\nu), \hat{f}(x,u,\nu)) \leq \epsilon. \quad (29)$$

Moreover the following complexity estimates hold, depending on \mathcal{X} and f :

- (i) **Efficient Case:** If \mathcal{X} is n -efficient for f , for some $n \in \mathbb{N}_+$ and f -normalized, then g is as in Theorem 20 (but with $p+a+2^{-1}n(n+1)$ and $2^{-1}n(n+1)$ in place of p and of m , respectively) and the right-hand side of (29) is $\kappa\sqrt{n(n+1)}\epsilon$, for a $\kappa > 0$ independent of m , n , and of ϵ .
- (ii) **General Case:** If \mathcal{X} is not efficient for f , then g has width $p+a+2^{-1}(n(n+1)+n(n+1))+2$ and depth recorded in Table 1 (but with $p+a+2^{-1}n(n+1)$ and $2^{-1}n(n+1)$ in place of p and m , respectively).

Remark 41 *The architecture of Corollary 39 is actually more general than deep Kalman filter of Krishnan et al. (2015) since it can process Gaussian measures with non-diagonal covariances.*

4.4.3 UNIVERSAL APPROXIMATION TO PROJECTIVE SHAPE SPACE

Since its introduction in Kendall (1984), various (pre-)shape spaces have appeared in the literature (e.g. Begelfor and Werman (2006), Goodall and Mardia (1999)) most of which are summarized in either of the monographs Bhattacharya and Bhattacharya (2012) and Dryden and Mardia (2016). In each case, the user seeks to filter out certain transformation k -tuples (called k -ads in the computer-vision literature). The major difference between these shape spaces is which transformations are filtered out and how they are filtered. In this section, we focus on the recently introduced *projective shape space* $P\Sigma_m^k$ of Mardia and Patrangenaru (2005) for two major reasons. First, its geometry is much more well-behaved than Kendall’s shape space and second, its structure provides a perfect example of how to utilize the entire computational graph of Figure 5.

We fix $k > m + 2$. An element of $P\Sigma_m^k$ is a *projective shape of an $m + 1$ -dimension k -ad*; which is defined as follows. Considers a k -ads $X \triangleq (x'_k)_{k=1}^K$, where each $x'_k \in \mathbb{R}^{m+1}$, which correspond to non-degenerate shapes; which, for Mardia and Patrangenaru (2005), means that $(x'_k)_{k=1}^K$ span \mathbb{R}^{m+1} and each $x'_k \neq 0$. The effect of scaling is then removed by setting $x_k \triangleq \frac{1}{\|x'_k\|} x'_k \in S^m$. One then applies the projection of $S^m \ni x \mapsto \{x, -x\} = [x] \in \mathbb{R}P^m$. Thus, the k -ad $\{x_k\}_{k=1}^K$ is sent to the element $([x_k])_{k=1}^K \in (\mathbb{R}P^m)^K$. Finally, all fractional linear transformations are filtered out of the k -ad by sending $([x_k])_{k=1}^K \in (\mathbb{R}P^m)^K$ to its *projective shape* defined by:

$$[X] \triangleq \{([Ax_1], \dots, [Ax_K]) : A \in \text{GL}_{m+1}\},$$

where GL_{m+1} is the set of all invertible $m + 1 \times m + 1$ -matrices. The space $P\Sigma_m^k$ is the manifold of all *projective shapes* constructed in this manner. In [Mardia and Patrangenaru \(2005\)](#), it is shown that $P\Sigma_k^m$ is a Riemannian manifold and a detailed description of its distance $d_{P\Sigma_k^m}$ is provided therein.

Though this description most clearly explains what a projective shape is, there are analytically simpler descriptions. Markedly, in ([Mardia and Patrangenaru, 2005](#), Proposition 2.2-2.3) a smooth bijection with smooth inverse $\rho : \prod_{k=1}^{K-m-2} \mathbb{R}P^m \xrightarrow{P} \Sigma_m^k$ is defined in closed-form via certain algebraic relations. Most notably for our context is the fact that ρ satisfies Assumption 8 (i) since ρ^{-1} is a well-define continuous map on all of $\prod_{k=1}^{K-m-2} \mathbb{R}P^m$ and, since ρ is a bijection, then it satisfies Assumption 8 (ii) via the trivial homotopy $H : (t, y) \in [0, 1] \times P\Sigma_m^k \mapsto y \in P\Sigma_m^k$. Note, ρ is also Lipschitz (with some constant $\kappa > 0$) since it is smooth and each $\mathbb{R}P^m$ is compact.

As a contextual interpretation of our approximation theorem, consider the following dimension-reduction task for computer vision. Suppose that we are provided with a dataset of high-resolution images $\mathcal{X} \subseteq \mathbb{R}^{P \times p}$ (P are the number of pixels in each image, k is the number of features described by each pixel (e.g. RGB, alpha, etc...)). We then want to extract the most essential $k \ll P$ pixels from each shape, and we want to compress the information in each pixel to some low-dimensional “deep features” $m \ll p$. Then, our objective would be to learn a function $f \in C(\mathbb{R}^{P \times p}, P\Sigma_m^k)$ which most efficiently performs this learning task according to some performance metric. In that case, Theorem 37 guarantees that any such “compression” function can be learned via the following deep neural models:

$$\hat{f} \triangleq \rho([\text{Exp}_{S^m, y_1} \circ g_1], \dots, [\text{Exp}_{S^m, y_k} \circ g_k]), \quad (30)$$

where $g_i \in \mathcal{NN}_{Pp, m}^\sigma$, y_i , Exp_{S^m, y_i} is defined in (15), and $S^m \ni x \mapsto [x] \in \mathbb{R}P^m$ is defined in Example 7 (for $i = 1, \dots, k$). The learning model of (30) is a universal projective shape analogue of the well-known “linear” regression models of [Davis et al. \(2010\)](#); [Thomas Fletcher P. \(2013\)](#) designed for Kendall’s pre-shape space.

Corollary 42 (Universal Approximator to Projective Shape Space $P\Sigma_m^k$) *Let $\epsilon > 0$, σ satisfy Condition 1, and $f \in C(\mathbb{R}^{Pp}, \Sigma P_m^k)$. Then, for every compact $K \subseteq \mathbb{R}^{P \times p}$ of diameter at-most $\frac{\pi}{2}$ there is an \hat{f} as in (30) satisfying:*

$$\max_{x \in K} d_{P\Sigma_m^k}(f(x), \hat{f}(x)) < \epsilon.$$

4.5 New Insights for “Classical” Euclidean Universal Approximation

Our main results provide a number of new insights into the behaviour of deep feedforward networks between classical Euclidean spaces. First, since \mathbb{R}^p and \mathbb{R}^m have everywhere 0 sectional curvature, then Corollary 12 implies that $\mathcal{U}_f(x) = \infty$ for every $f \in C(\mathbb{R}^p, \mathbb{R}^m)$ and every $x \in \mathbb{R}^p$. As shown in [Jost \(2017\)](#), for every $x \in \mathbb{R}^d$ we have the identities:

$$\text{Exp}_{\mathbb{R}^d, x}(y) = y + x \text{ and } \text{Exp}_{\mathbb{R}^d, x}^{-1}(y) = y - x.$$

Thus, Theorem 9 reduces to a quantitative version of the (*qualitative*) *universal approximation theorem for deep and narrow feedforward networks* derived in [Kidger and Lyons \(2020\)](#).

Corollary 43 (Quantitative Deep and Narrow Universal Approximation Theorem) *Fix $f \in C(\mathbb{R}^p, \mathbb{R}^m)$ and suppose that $\sigma \in C(\mathbb{R})$ is non-affine and piecewise linear. Then, for every compact $K \subset \mathbb{R}^p$ and every $\epsilon > 0$ there exists a DNN $g \in \mathcal{NN}_{p, m, p+m+2}^\sigma$ satisfying:*

$$\sup_{x \in X} \|g(x) - f(x)\| < \epsilon.$$

Moreover, the g ’s depth is recorded in Table 1.

Our results on efficient dataset imply the following extension of the dimension-free approximation results of [Yarotsky and Zhevnerchuk \(2020\)](#).

Corollary 44 *Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ and let $\mathcal{X} \subseteq [0, 1]^p$ be an n -efficient dataset for f . For each $\epsilon > 0$, there is a $W \in \mathbb{N}_+$, a $g \in \mathcal{NN}_{p,m,W}^\sigma$, and a $\kappa > 0$ not depending on ϵ , p , or on m , such that:*

$$\sup_{x \in K} \|f(x) - \hat{f}(x)\| \leq \kappa m^{\frac{1}{2}} \epsilon. \quad (31)$$

Moreover, g satisfies the following sub-exponential complexity estimates:

(i) **Width:** satisfies $m \leq W \leq m(4p + 10)$,

(ii) **Depth:** of order $\mathcal{O}\left(m + m\epsilon^{\frac{2p}{3(np+1)} - \frac{p}{np+1}}\right)$,

(iii) **Number of trainable parameters:** is of order $\mathcal{O}\left(m(m^2 - 1)\epsilon^{-\frac{2p}{3(np+1)}}\right)$.

Corollary 44 is a strict extension of the analogous result of [Yarotsky and Zhevnerchuk \(2020\)](#). This is because, Propositions 59 and 19 guaranteed that every finite dataset is efficient for every function (and not only highly smooth ones as considered in [Yarotsky and Zhevnerchuk \(2020\)](#)). Moreover, every dataset is efficient for functions in $C^{nd,1}([0, 1]^p, \mathbb{R}^m)$.

The first of these remarks gives a concrete and general explanation of why deep ReLU networks perform so well in practice. Namely, they can always avoid the curse of dimensionality for “real-world datasets”. We emphasise that the proof of this claim is non-trivial and relies on the notorious *Whitney(-Fefferman) Extension Theorem*. We emphasize that there is no continuity assumption on the target function in the next result (let-alone any smoothness assumptions).

Corollary 45 (ReLU DNN Approximation is Not Cursed on “Real-World” Datasets) *If $\mathcal{X} \subseteq [0, 1]^p$ is a “real-world dataset” (i.e.: finite and non-empty) and $\sigma : x \mapsto \max\{0, x\}$. For every function $f : \mathbb{R}^p \rightarrow \mathbb{R}^m$ there is a $\kappa > 0$ such that, for every $\epsilon > 0$ there is a $g \in \mathcal{NN}_{p,m,W}^\sigma$ satisfying:*

$$\max_{x \in \mathcal{X}} \|f(x) - g(x)\| \leq \kappa m^{\frac{1}{2}} \epsilon.$$

Moreover, g satisfies the following sub-exponential complexity estimates:

(i) **Width:** satisfies $m \leq W \leq m(4p + 10)$,

(ii) **Depth:** of order $\mathcal{O}\left(m + m\epsilon^{\frac{2p}{3(np+1)} - \frac{p}{np+1}}\right)$,

(iii) **Number of trainable parameters:** is of order $\mathcal{O}\left(m(m^2 - 1)\epsilon^{-\frac{2p}{3(np+1)}}\right)$.

We now summarize the contributions made in this paper.

Conclusion

This paper introduces a general and flexible “*differentiable geometric deep learning framework*” for explicitly building universal approximators of a feedforward type between any differentiable manifold. Our results are quantitative; they provide a (non-trivial) lower bound on the maximum radius in which our approximation guarantees could hold, and they estimate the complexity of the involved deep neural models’ depth and width. We derive dimension-free approximation rates on any efficient dataset. This novel concept strictly expand the efficient approximation results for smooth functions with Lipschitz higher-order partial derivatives. We also prove that it is impossible to obtain global universal approximation results in the general non-Euclidean context utilizing a

topological obstruction result (which always vanishes in the Euclidean setting, and more generally for pairs of Cartan-Hadamard input/output spaces).

We use our theory to derive quantitative and efficient universal approximation theorems for a host of commonly implemented differentiable geometric deep learning architectures. We show how popular geometric regression models could be simply extended to universal GDL models.

As a final application, we show that our results imply multiple new insights even in the classical Euclidean setting. These include a quantitative version of the approximation theorem of [Kidger and Lyons \(2020\)](#) for deep and narrow feedforward networks and a general guarantee that deep ReLU networks always beat the curse of dimensionality when approximating any target function on any real-world dataset.

We believe that the versatility, simplicity, and scope of the proposed framework provide a satisfactory differentiable geometric deep learning theory since, as we have seen, any continuous function between differentiable manifolds can be universally approximated via our *differentiable geometric deep learning* models.

Future Work

The work developed in the current project leads to at least two future GDL research questions, which we will be working on shortly. The first is a dynamic follow-up to this project, where we would like to use the theory developed herein to build universal RNNs and reservoir computers between differentiable manifolds. The second follow-up problem is the treatment of non-differentiable input and output spaces. In that follow-up project, we envision building on our theory by extending it to a GDL framework capable of processing non-differentiable input and output spaces such as graphs, trees, and hierarchical structures.

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This appendix complements the article’s main body by providing additional background material to the mathematical background used throughout the paper, and by providing detailed proofs of each of the results.

Appendix A. Additional Background

To help the paper be as self-contained as possible, this appendix contains some relevant results and background from constructive approximation theory and from algebraic topology.

A.1 Bernstein polynomials and a quantitative shallow universal approximation theorem

The classical result on neural networks is the so-called universal approximation theorem, mentioned in the introduction. It is at the core of our quantitative estimates, so we restate it here. Mathematically, a single-hidden layer neural network with activation function σ can be written, for $x \in \mathbb{R}^p$,

$$\sum_{i=1}^n c_i \sigma(\langle w_i, x \rangle - \theta_i) \tag{32}$$

for some $w_i \in \mathbb{R}^p$, $c_i, \theta_i \in \mathbb{R}$. Here, the number of terms n is the sum corresponds to the number of neurons in the hidden layer. We then have the following.

Theorem 46 (*Pinkus, 1999a, Theorem 3.1*) *Let $\sigma \in \mathcal{C}(\mathbb{R})$. Then*

$$\mathcal{N}(\sigma) := \left\{ \sum_{i=1}^n c_i \sigma(\langle w_i, x \rangle - \theta_i) : n \in \mathbb{N}, w_i \in \mathbb{R}^p, c_i, \theta_i \in \mathbb{R} \right\}$$

is dense in $\mathcal{C}(\mathbb{R}^p)$ in the topology of uniform convergence on compact sets if and only if σ is not a polynomial.

The first step to prove theorem 46 is to use the Stone–Weierstrass theorem to approximate the target function by a polynomial. This multivariate polynomial is then approximated by a shallow network. Therefore, in order to derive quantitative estimates for theorem 46, we need to obtain a rate of convergence for the Stone–Weierstrass theorem. For continuous functions defined on \mathbb{R} , a proof of this theorem relies on Bernstein polynomials, thereby providing an explicit rate of convergence. In higher dimensions, a rate of convergence can be obtained using a multi-dimensional version of these Bernstein polynomials, that we introduce next. We use the following notation. For $x \in [0, 1]$, $n, k \in \mathbb{N}$ such that $n \geq k$, we denote

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

Definition 47 *The multidimensional Bernstein operator $B_n : \mathcal{C}([0, 1]^p, \mathbb{R}) \rightarrow \mathcal{C}([0, 1]^p, \mathbb{R})$ is defined by, for $x = (x_1, \dots, x_p) \in \mathbb{R}^p$,*

$$B_n(f, x) := \sum_{k_1=0}^n \cdots \sum_{k_p=0}^n f\left(\frac{k_1}{n}, \dots, \frac{k_p}{n}\right) p_{n,k_1}(x_1) \cdots p_{n,k_p}(x_p).$$

$B_n(f, \cdot)$ is called the multivariate Bernstein polynomial associated to f .

Using these multidimensional Bernstein operators, we obtain the following quantitative version of the Stone–Weierstrass theorem for real-valued functions defined on \mathbb{R}^p . The rate of convergence depends on the modulus of continuity of the target function $f \in \mathcal{C}([0, 1]^p, \mathbb{R})$ which is defined as, for $\epsilon > 0$,

$$\omega(f, \epsilon) = \sup\{|f(x) - f(y)| : \|x - y\| \leq \epsilon\}.$$

Proposition 48 *Let $f \in \mathcal{C}([0, 1]^p, \mathbb{R})$. Then, for all $n \in \mathbb{N}$, $h > 0$:*

$$\|B_n(f) - f\|_\infty \leq \left(1 + \frac{p}{4}\right) \omega\left(f, \frac{1}{\sqrt{n}}\right).$$

Proposition 48 is a consequence of two theorems on properties of a certain Bochner-type integral and of the convexity of $[0, 1]^p$. The first theorem, that we state below, defines this Bochner-type integral.

Theorem 49 (*Paltanea, 2004, Theorem 6.2.1*) *Let μ be a Borel positive measure on $[0, 1]^p$ such that $\mu([0, 1]^p) > 0$. For any $f \in \mathcal{C}([0, 1]^p, \mathbb{R})$, there is a unique $b \in \mathbb{R}$ having the following property. For any $\epsilon > 0$, there exists $\delta > 0$ such that for any partition $\{D_1, \dots, D_m\}$ of $[0, 1]^p$ with $\mu(D_i) \leq \delta$ and for any choice of $x_i \in [0, 1]^p$, $i = 1, \dots, m$, we have*

$$\left| b - \sum_{i=1}^m f(x_i) \mu(D_i) \right| \leq \epsilon.$$

We denote $b := F_{\mathbb{R}}(f)$ and $F_{\mathbb{R}} : \mathcal{C}([0, 1]^p, \mathbb{R}) \rightarrow \mathbb{R}$ is a Bochner-type integral.

The second relevant result to prove Proposition 48 bounds the difference between a point evaluation of a function and the value of its Bochner integral.

Theorem 50 (*Paltanea, 2004, Theorem 6.2.3*) For any $f \in \mathcal{C}([0, 1]^p, \mathbb{R})$, $x \in [0, 1]^p$ and $h > 0$,

$$|F_{\mathbb{R}}(f) - f(x)| \leq |f(x)| |F_{\mathbb{R}}(e_0) - 1| + (F_{\mathbb{R}}(e_0) + h^{-2}F_{\mathbb{R}}(\|\cdot - x\|^2))\omega(f, h) \quad (33)$$

where $e_0(x) = 1$ for all $x \in [0, 1]^p$.

Proof [Proof of Theorem 50] The proof of this theorem relies on elementary properties of the Bochner-type integral $F_{\mathbb{R}}$ constructed in theorem 49. In particular, this operator is linear and positive. For $f \in \mathcal{C}([0, 1]^p, \mathbb{R})$, $x \in [0, 1]^p$, $h > 0$, we then have

$$\begin{aligned} |F_{\mathbb{R}}(f) - f(x)| &\leq |F_{\mathbb{R}}(f) - F_{\mathbb{R}}(f(x)e_0)| + |F_{\mathbb{R}}(f(x)e_0) - f(x)| \\ &= |F_{\mathbb{R}}(f - f(x))| + |f(x)| |F_{\mathbb{R}}(e_0) - 1| \quad \text{by linearity of } F_{\mathbb{R}} \\ &\leq F_{\mathbb{R}}(e_0 + h^{-2}\|\cdot - x\|_{\mathbb{R}^p})\omega(f, h) + |f(x)| |F_{\mathbb{R}}(e_0) - 1| \\ &\leq \left(F_{\mathbb{R}}(e_0) + h^{-2}F_{\mathbb{R}}(\|\cdot - x\|_{\mathbb{R}^p}) \right) \omega(f, h) + |f(x)| |F_{\mathbb{R}}(e_0) - 1|. \end{aligned}$$

■

We can now turn to the proof of proposition 48. The idea is to define, for $x \in [0, 1]^p$ fixed, a measure on $[0, 1]^p$ for which the Bochner integral of $f \in \mathcal{C}([0, 1]^p, \mathbb{R})$ with respect to this measure is precisely the Bernstein polynomial associated to f (see definition 47). Then, it suffices to compute the right-hand side of inequality (33) to obtain the stated convergence result.

Proof [Proof of Proposition 48] For $n \in \mathbb{N}$, $x \in [0, 1]^p$, we define the positive Borel measure on $[0, 1]^p$

$$B_n(\cdot, x) := \sum_{k_1=0}^n \cdots \sum_{k_p=0}^n \delta_{\frac{k_1}{n}, \dots, \frac{k_p}{n}} p_{n, k_1}(x_1) \cdots p_{n, k_p}(x_p)$$

where δ is the Dirac measure. By example 6.2.1 in Paltanea (2004), for all $f \in \mathcal{C}([0, 1]^p, \mathbb{R})$, $x \in [0, 1]^p$, it holds that

$$(B_n)_{\mathbb{R}}(e_0, x) = 1, \quad \text{and} \quad (B_n)_{\mathbb{R}}(\|\cdot - x\|^2, x) = \sum_{j=1}^p \frac{x_j(1-x_j)}{n}.$$

Therefore, by theorem 50, for all $f \in \mathcal{C}([0, 1]^p, \mathbb{R})$, $x \in [0, 1]^p$, $h > 0$,

$$|B_n(f, x) - f(x)| \leq \left(1 + h^{-2} \sum_{j=1}^p \frac{x_j(1-x_j)}{n} \right) \omega(f, h).$$

Thus, taking $h = n^{-1/2}$ and bounding $x_j(1-x_j)$ by $1/4$, we obtain

$$\|B_n(f) - f\|_{\infty} \leq \left(1 + \frac{p}{4} \right) \omega(f, \frac{1}{\sqrt{n}}).$$

■

A.2 Riemannian Geometric Background

Fix $p \in \mathbb{N}$. Broadly speaking, a p -dimensional *Riemannian manifold* is a topological space \mathcal{X} which has a locally analogous geometry to Euclidean space in that it has locally interrelated notions of distance, angle, and volume, all of which are locally comparable to their analogs in \mathbb{R}^p .

We briefly build up Riemannian manifolds from more elementary geometric objects, beginning with *smooth manifolds*. A smooth manifold \mathcal{X} , introduced in [Riemann \(1854\)](#), is a topological space on which a familiar differential calculus may be built, analogous to \mathbb{R}^p . Since the derivative is a purely local object of any function, we only require that \mathcal{X} can locally be identified with \mathbb{R}^p . This local identification is achieved through a system of open subsets $\{U_\alpha\}_{\alpha \in A}$ of \mathcal{X} which are identified with open subsets of \mathbb{R}^p via continuous bijections $\phi_\alpha : U_\alpha \rightarrow \mathbb{R}^p$ each of which has a continuous inverse. For any $\alpha, \beta \in A$, the functions $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\alpha \cap U_\beta)$ are defined between subsets of the familiar Euclidean space \mathbb{R}^p . Therefore we enforce a well-defined local calculus on all of \mathcal{X} by requiring that each of these maps is infinitely differentiable. The collection $\{(U_\alpha, \phi_\alpha)\}_\alpha$ is called an *atlas* and each (U_α, ϕ_α) therein is called a *coordinate patch*.

Since we would like to linearize functions defined on \mathcal{X} via their derivative, we need to extend the notion of a tangent line from Calculus to \mathcal{X} to a collection of vector spaces lying tangential to the points of \mathcal{X} . The construction of a tangent space begins with the definition of a *smooth function* $f : \mathcal{X} \rightarrow \mathbb{R}$, which is a continuous function for which each $f \circ \phi_\alpha^{-1}$ is infinitely differentiable. The set of smooth functions on \mathcal{X} is denoted by $C^\infty(\mathcal{X})$. The derivative of a differentiable function $f : \mathbb{R}^p \rightarrow \mathbb{R}^m$ at some $x \in \mathbb{R}^p$ is a linear map $df_x : \mathbb{R}^p \rightarrow \mathbb{R}^m$ which approximates f via $f(x + \Delta) \approx f(x) + df_x \Delta + o(\Delta)$. Furthermore, the linearization operation at x , sending $f \mapsto df_x$, is characterized by the product rule

$$d(fg)_x = f(x)dg_x + df_x g(x). \quad (34)$$

Thus, the set of all tangent vectors at x to some differentiable function is identified with the linear maps from $C^\infty(\mathbb{R}^p)$ to \mathbb{R} . This perspective is convenient, since the set of tangent vectors at any $x \in \mathcal{X}$, denoted by $T_x(\mathcal{X})$, is the p -dimensional vector space of linear maps from $C^\infty(\mathcal{X})$ to \mathbb{R} . In many situations the vector space $T_x(\mathcal{X})$ admits a simple description and, whenever convenient, it is identified with \mathbb{R}^p .

Tangent spaces allow us to define an intrinsic notion of distance on \mathcal{X} . The description begins with *vector fields*; these can be understood as a rule which smoothly assigns a vector at each point of \mathcal{X} and they are defined as linear maps $X : C^\infty(\mathcal{X}) \rightarrow C^\infty(\mathcal{X})$ satisfying the product-rule $X(fg) = f(X(g)) + g(X(f))$. Vector fields allow us to define *Riemannian metrics* on \mathcal{X} , these are families of inner products $g \triangleq (g_x)_{x \in \mathcal{X}}$ with each g_x defined on $T_x(\mathcal{X})$ such that $x \mapsto g_x(X|_x, Y|_x)$ is a smooth map, for every pair of vector fields X and Y on \mathcal{X} . Together, (\mathcal{X}, g) define a *Riemannian manifold*, which we denote by \mathcal{X} when the context is clear. Riemannian metrics are of interest, since they induce an intrinsic distance $d_{\mathcal{X}}$ on \mathcal{X} which is locally analogous to the Euclidean distance, and represents the length of the shortest tractable path between any two points $x_1, x_2 \in \mathcal{X}$ via

$$d_{\mathcal{X}}(x_1, x_2) \triangleq \inf \left\{ \int_0^1 \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))} dt : \gamma(0) = x_1, \gamma(1) = x_2, \text{ and } \gamma \text{ is piece-wise smooth} \right\},$$

where $\dot{\gamma}$ denotes the derivative of the curve γ and, by definition, it exists for almost all $0 \leq t \leq 1$.

If the intrinsic distance $d_{\mathcal{X}}$ defines a complete metric on \mathcal{X} , then there is a standard open neighborhood about any $x \in \mathcal{X}$ which can be identified with a Euclidean ball. This is because, together the results of [Rinow \(1964\)](#) and of [Lindelöf \(1894\)](#) guarantee that for any $x \in \mathcal{X}$ and any $u \in T_x(\mathcal{X})$ of sufficiently small norm, there exists a unique smooth curve γ originating at x , with initial velocity $\dot{\gamma}(0) = u \in T_x(\mathcal{X})$, and of minimal length, i.e.: $d_{\mathcal{X}}(x, \gamma(T)) = \int_0^T \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))} dt$, for all $0 \leq T \leq 1$. For any $x \in \mathcal{X}$, the least upper-bound on the norm of $u \in T_x(\mathcal{X})$ guaranteeing the existence of such a γ is called the *injectivity radius* of \mathcal{X} at x , and it is denoted by $\text{inj}_{\mathcal{X}}(x)$. The injectivity radius is key in our analysis since it allows us to linearizing \mathcal{X} about any point x while preserving the intrinsic distance between x and points on \mathcal{X} near it. We do this through the

Riemannian exponential map at $x \in \mathcal{X}$, denoted by $\text{Exp}_{\mathcal{X},x}$, that sends any tangent vector u at x to $\gamma(1)$, where γ is the distance-minimizing curve with initial conditions $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. It is a well-defined homeomorphism from $B_{\mathbb{R}^p}(0, \text{inj}_{\mathcal{X}}(x))$ onto $B_{\mathcal{X}}(x, \text{inj}_{\mathcal{X}}(x))$, making it a natural choice for a local feature and readout map, as it additionally preserves the distance between x and any point $y \in B_{\mathbb{R}^p}(0, \text{inj}_{\mathcal{X}}(x))$ through the *radial isometry condition* $d_{\mathcal{X}}(x, y) = \left\| \text{Exp}_{\mathcal{X},x}^{-1}(y) \right\|$.

Two additional analogues between Euclidean space and Riemannian manifolds, which we use at different stages of our analysis, are its intrinsic volume and its orientation. The *intrinsic volume* $\text{Vol}_{\mathcal{X}}(B)$ of any Borel set $B \subseteq U_{\alpha}$ in the coordinate patch $(U_{\alpha}, \phi_{\alpha})$ is $\int_{x \in \phi_{\alpha}(B)} \sqrt{g \circ \phi_{\alpha}^{-1}} dx$. A Riemannian manifold \mathcal{X} is orientable if it is impossible to smoothly move a three-dimensional figure along \mathcal{X} in such a way that the moving eventually results in the figure being flipped, rigorously, \mathcal{X} must admit an atlas $\{(U_{\alpha}, \phi_{\alpha})\}_{\alpha}$ where each ϕ_{α} has positive Jacobian determinant. We also denote the set of 2 dimensional planes attached smoothly across \mathcal{X} by $G_{p,2}(\mathcal{X})$. For more details on Riemannian geometry we refer the reader to [Jost \(2017\)](#).

Appendix B. Proofs

The remainder of this appendix is devoted to the proofs of our paper's main results. We emphasize that, the order in which the proofs are derived differs from the order they were exposed in the paper's main body. We chose this order so as to simplify both the paper and the appendix's flow.

B.1 Technical Lemmas

This appendix contains proof of the paper's results as well as any relevant technical lemmas. The proof of [Theorem 9](#) relies of the following two localization lemmas.

Lemma 51 *Let \mathcal{X} and \mathcal{Y} a complete connected Riemannian manifolds satisfying [Assumption 2](#), of respective dimension p and m , and let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a continuous function. For any $x \in \mathcal{X}$, if*

$$0 < \delta < \min \left\{ \delta(\mathcal{X}, x, k_{\mathcal{X}}^*), \omega^{-1} \left(f, \delta(\mathcal{Y}, f(x), k_{\mathcal{Y}}^*) \right) \right\}$$

then the following hold:

- (i) $\text{Exp}_{\mathcal{X},x}^{-1} : \overline{B_{\mathcal{X}}(x, \delta)} \rightarrow \mathbb{R}^p$ is a diffeomorphism onto its image. In particular, it is Lipschitz with constant $L_{\mathcal{X},x} > 0$,
- (ii) $\text{Exp}_{\mathcal{Y},f(x)}^{-1} : f(\overline{B_{\mathcal{X}}(x, \delta)}) \rightarrow \text{Exp}_{\mathcal{Y},f(x)}^{-1} \left(f(\overline{B_{\mathcal{X}}(x, \delta)}) \right) \subseteq \mathbb{R}^m$ is a diffeomorphism onto its image. In particular, $\text{Exp}_{\mathcal{Y},f(x)}$ is Lipschitz on $\text{Exp}_{\mathcal{Y},f(x)}^{-1} \left(f(\overline{B_{\mathcal{X}}(x, \delta)}) \right)$ with constant $L_{\mathcal{Y},f(x)} > 0$.

In particular, we have the following estimate:

$$0 < \min \left\{ \delta(\mathcal{X}, x, k_{\mathcal{X}}^*), \omega^{-1} \left(f, \delta(\mathcal{Y}, f(x), k_{\mathcal{Y}}^*) \right) \right\} \leq \min \left\{ \delta(\mathcal{X}, x, k_{\mathcal{X}}^*), \omega^{-1} \left(f, \delta(\mathcal{Y}, f(x), k_{\mathcal{Y}}^*) \right) \right\}.$$

Proof By [Assumption 2](#) (i) we have the following finite bound on the Riemannian curvature $K_{\mathcal{X}}$ of \mathcal{X}

$$0 \leq K(\mathcal{X}) \triangleq \sup_{\pi_x(u,v): x \in \mathcal{X}, \pi_x(u,v) \in G_{p,2}(\mathcal{X})} |K_{\mathcal{X}}(\pi_x(u, v))| < \infty.$$

Since \mathcal{X} is a complete Riemannian manifold, then ([Cheeger et al., 1982](#), [Theorem 4.7](#)) implies the following lower-bound on the injectivity radius at any $x \in \mathcal{X}$

$$\text{inj}(x) > r \frac{\text{Vol}_{\mathcal{X}}(B_{\mathcal{X}}(x, r))}{\text{Vol}_{\mathcal{X}}(B_{\mathcal{X}}(x, r)) + \text{Vol}_{T_x(\mathcal{X})}(B_{T_x(\mathcal{X})}(0, 2r))}, \quad (35)$$

for any $0 < r < k_{\mathcal{X}}^*$. In particular,

$$\text{inj}(x) \geq \delta(\mathcal{X}, x, k_{\mathcal{X}}^*). \quad (36)$$

By Assumption 2 (ii), we have that $\text{Vol}_{\mathcal{X}}(B_{\mathcal{X}}(x, r)) > 0$ for any $r > 0$ and therefore the right-hand side of (35) is non-zero. Hence, (36) refines to

$$\text{inj}(x) \geq \delta(\mathcal{X}, x, k_{\mathcal{X}}^*) > \delta > 0. \quad (37)$$

By (Jost, 2017, Corollary 1.7.1), the map $\text{Exp}_{\mathcal{X},x}^{-1}$ is a diffeomorphism from $B_{\mathbb{R}^p}(0, \text{inj}(x))$ onto $\text{Exp}_{\mathcal{X},x}(B_{\mathbb{R}^p}(0, \text{inj}(x)))$. Moreover, since $\text{Exp}_{\mathcal{X},x}$ is a radial-isometry (see the discussion following (Jost, 2017, Corollary 1.4.2)) then $\text{Exp}_{\mathcal{X},x}(B_{\mathbb{R}^p}(0, \tilde{\delta})) = B_{\mathcal{X}}(x, \tilde{\delta})$ for every $0 < \tilde{\delta} \leq \text{inj}(x)$. Since $\delta < \text{inj}(x)$ then,

$$\text{Exp}_{\mathcal{X},x}^{-1} \big|_{\overline{B_{\mathcal{X}}(x, \delta)}} : \overline{B_{\mathcal{X}}(x, \delta)} \rightarrow \overline{B_{\mathbb{R}^p}(0, \delta)}, \quad (38)$$

is a diffeomorphism. Since $\text{Exp}_{\mathcal{X},x}^{-1}$ is a diffeomorphism then it is in particular Lipschitz with constant $L_{\mathcal{X},x} > 0$. Thus, (i) holds.

Next, let $\tilde{x} \in B_{\mathcal{X}}(x, \delta)$. Then, by the definition of the modulus of continuity of f , by its monotonicity, and since $\omega(f, \delta) < \delta(\mathcal{Y}, f(x), k_{\mathcal{Y}}^*)$ we compute

$$\begin{aligned} d_{\mathcal{Y}}(f(\tilde{x}), f(x)) &\leq \omega(f, d_{\mathcal{X}}(\tilde{x}, x)) \\ &< \omega(f, \delta) \\ &\leq \delta(\mathcal{Y}, f(x), k_{\mathcal{Y}}^*); \end{aligned} \quad (39)$$

where the last inequality follows from (Embrechts and Hofert, 2013, Proposition 1 (5)). In particular, $f(\overline{B_{\mathcal{X}}(x, \delta)}) \subseteq \overline{B_{\mathcal{Y}}(f(x), \delta(\mathcal{Y}, f(x), k_{\mathcal{Y}}^*))}$.

As in the proof of (i), under Assumptions 2 (i) and (ii), (Cheeger et al., 1982, Theorem 4.7) implies that

$$\text{inj}_{\mathcal{Y}}(f(x)) \geq \delta(\mathcal{Y}, f(x), k_{\mathcal{Y}}^*) > 0,$$

and therefore (Jost, 2017, Corollaries 1.7.1 and 1.4.2) implies that $\text{Exp}_{\mathcal{Y},f(x)}$ is a radially-isometric diffeomorphism from $B_{\mathbb{R}^m}(0, \delta(\mathcal{Y}, f(x), k_{\mathcal{Y}}^*))$ onto $B_{\mathcal{Y}}(f(x), \delta(\mathcal{Y}, f(x), k_{\mathcal{Y}}^*))$. Moreover, as before, since $f(\overline{B_{\mathcal{X}}(x, \delta)})$ is a compact subset of $B_{\mathcal{Y}}(f(x), \text{inj}_{\mathcal{Y}}(f(x)))$ then

$\text{Exp}_{\mathcal{Y},f(x)} : f(\overline{B_{\mathcal{X}}(x, \delta)}) \rightarrow \mathbb{R}^m$ is Lipschitz with some constant $L_{\mathcal{Y},f(x)} > 0$. This gives (ii). \blacksquare

So as not to disrupt the appendix's overall flow, we maintain the notation introduced in the proof of Lemma 51 within the next Lemma's statement and its proof.

Lemma 52 *Let \mathcal{X} be a complete connected Riemannian manifold, $f : \mathcal{X} \rightarrow \mathcal{Y}$. For any $x \in \mathcal{X}$, we denote the Lipschitz constant of $\text{Exp}_{\mathcal{X},x}$ on $\overline{B_{\mathcal{X}}(0, \text{inj}_{\mathcal{X}}(x))}$ by $L_{\mathcal{X},x}^{-1}$ and we use $L_{\mathcal{Y},f(x)}^{-1}$ to denote the Lipschitz constant of $\text{Exp}_{\mathcal{Y},f(x)}^{-1}$ on $\overline{B_{\mathcal{Y}}(0, \text{inj}_{\mathcal{Y}}(f(x)))}$. If it holds that:*

$$0 < \delta < \min \{ \text{inj}_{\mathcal{X}}(x), \omega^{-1}(f, \text{inj}_{\mathcal{Y}}(f(x))) \}$$

then, on the compact set $\overline{B_{\mathcal{X}}(x, \delta)}$ the map f can be represented as

$$f = \text{Exp}_{\mathcal{Y},f(x)} \circ \tilde{f} \circ \text{Exp}_{\mathcal{X},x}^{-1} \quad (40)$$

where $\tilde{f} : \text{Exp}_{\mathcal{X},x}^{-1}(\overline{B_{\mathcal{X}}(x, \delta)}) \rightarrow \mathbb{R}^m$ is defined by:

$$\tilde{f} \triangleq \text{Exp}_{\mathcal{Y},f(x)}^{-1} \circ f \circ \text{Exp}_{\mathcal{X},x}. \quad (41)$$

Furthermore, if f is continuous, then so is \tilde{f} and its modulus of continuity $\omega(\tilde{f}, \cdot)$ is given by

$$\omega(\tilde{f}, \epsilon) = L_{\mathcal{Y},f(x)}^{-1} \omega(f, L_{\mathcal{X},x}^{-1} \epsilon). \quad (42)$$

Proof By Lemma 51 the map \tilde{f} is well-defined and continuous. Furthermore, by the same result, representation (40) holds since $\text{Exp}_{\mathcal{Y},f(x)}^{-1}$ and $\text{Exp}_{\mathcal{X},x}^{-1}$ are diffeomorphisms on $f\left(\overline{B_{\mathcal{X}}(x,\delta)}\right)$ and on $B_{\mathcal{X}}(x,\delta)$, respectively.

Lastly, we compute the modulus of continuity of \tilde{f} . By (42) and the fact that the modulus of continuity of a composition of uniformly continuous functions is equal to the composition of the moduli of continuity of the involved uniformly continuous functions, we have that

$$\begin{aligned}\omega(\tilde{f}, \epsilon) &= \omega\left(\text{Exp}_{\mathcal{Y},f(x)}^{-1} \circ f \circ \text{Exp}_{\mathcal{X},x}, \epsilon\right) \\ &= \omega\left(\text{Exp}_{\mathcal{Y},f(x)}^{-1}, \epsilon\right) \circ \omega(f, \epsilon) \circ \omega\left(\text{Exp}_{\mathcal{X},x}, \epsilon\right) \\ &= L_{\mathcal{Y},f(x)}^{-1} \omega\left(f, L_{\mathcal{X},x}^{-1} \epsilon\right).\end{aligned}$$

■

Next, we begin by deriving our depth estimates for deep and narrow feedforward networks from \mathbb{R}^p to \mathbb{R}^m . The result will subsequently be combined with the above lemmas to derive our controlled approximation results between more general non-Euclidean spaces.

B.2 Depth estimates for deep and narrow feedforward networks

This section is devoted to the proof of the quantification of the approximation results of Kidger and Lyons (2020). More precisely, given a prespecified error $\epsilon > 0$, we instigate how deep a neural network $g \in \mathcal{NN}_{p,m,p+m+2}^\sigma$ should be in order to approximate within the margin error a continuous function $f : K \rightarrow \mathbb{R}^m$ where $K \subset \mathbb{R}^p$ is compact. This quantitative proposition is at the core of all our quantitative estimates, even when we consider non-Euclidean input and output spaces.

Proposition 53 *Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be an activation function satisfying assumption 1. Let $K \subset \mathbb{R}^p$ be a compact set and let $f \in \mathcal{C}(K, \mathbb{R}^m)$. Then, for any $\epsilon > 0$, there exists $g \in \mathcal{NN}_{p,m,p+m+2}^\sigma$ such that $\|f - g\|_\infty \leq \epsilon$. Moreover:*

(i) *if σ is infinitely differentiable and non-polynomial, then the depth of g is of order*

$$O\left(m(\text{diam}K)^{2p}\left(\omega^{-1}\left(f, \frac{\epsilon}{\left(1 + \frac{p}{4}\right)m}\right)\right)^{-2p}\right) \quad (43)$$

(ii) *if σ is non-polynomial, then the depth of g is of order*

$$\begin{aligned}O\left(m(\text{diam}K)^{2p}\left(\omega^{-1}\left(f, \frac{\epsilon}{2m\left(1 + \frac{p}{4}\right)}\right)\right)^{-2p}\right. \\ \left.\left(\omega^{-1}\left(\sigma, \frac{\epsilon}{2Bm\left(2^{\text{diam}K^2\left[\omega^{-1}\left(f, \frac{\epsilon}{2m\left(1 + \frac{p}{4}\right)}\right)]^{-2+1} - 1\right)}\right)}\right)^{-1}\right)\right) \quad (44)\end{aligned}$$

for some $B > 0$ depending on f .

(iii) *if σ is a non-affine polynomial, then, if we allow an extra neuron on each layer of g , the depth of g is of order*

$$O\left(m(p+m)(\text{diam}K)^{4p+2}\left(\omega^{-1}\left(f, \frac{\epsilon}{\left(1 + \frac{p}{4}\right)m}\right)\right)^{-4p-2}\right). \quad (45)$$

Remark 54 *In point 3 of proposition 53, we have chosen to allow an extra neuron for clarity. However, depth estimates in the case where no extra neuron is allowed are derived in section B.2.5.*

B.2.1 AN EXTENSION RESULT

The depth estimates of Proposition 53 are derived in the case where the function f being approximated is defined on $[0, 1]^p$. Indeed, on such a domain of definition, the multivariate Bernstein polynomials introduced in definition 47 are well-defined and can be used to approximate each component of f . Restricting our analysis to functions defined on $[0, 1]^p$ is enough to derive estimates for functions defined on an arbitrary compact set K . Indeed, in this case, the function $f \in \mathcal{C}(K, \mathbb{R}^m)$ being approximated can be extended to the whole of \mathbb{R}^p in such a way that the extension preserves its modulus of continuity. By a simple change of variables, this extension can be considered on $[0, 1]^p$. Before stating our extension proposition, we introduce the following subadditive modulus of continuity, on which the extension relies.

Definition 55 *The concave majorant of $\omega(f, \cdot)$ is defined by, for $t > 0$,*

$$\omega_c(f, t) := \inf\{\alpha t + \beta : \omega(f, s) \leq \alpha s + \beta \ \forall s \in \mathbb{R}_+\}.$$

The subadditivity of $\omega_c(f, \cdot)$ is the key to prove the following extension proposition. This proposition will allow us to leverage depth estimates for functions defined on $[0, 1]^p$ to depth estimates for functions defined on an arbitrary compact set $K \subset \mathbb{R}^p$. It is a slightly different version of (McShane, 1934, Corollary 2).

Proposition 56 *Let $K \subset \mathbb{R}^p$ be a compact set. Let $f : K \rightarrow \mathbb{R}$ be a continuous function. Then f can be extended to \mathbb{R}^p by setting, for $x \in \mathbb{R}^p$,*

$$F(x) := \frac{1}{2} \sup_{y \in K} \{f(y) - \omega_c(f, \|x - y\|)\}. \quad (46)$$

Moreover, F preserves the modulus of continuity $\omega(f, \cdot)$, that is

$$\forall x, y \in \mathbb{R}^p, |F(x) - F(y)| \leq \omega(f, \|x - y\|). \quad (47)$$

We now illustrate why this proposition allows us to leverage our results for functions defined on $[0, 1]^p$ to functions defined on an arbitrary compact set $K \subset \mathbb{R}^p$. Let $K \subset \mathbb{R}^p$ be a compact set and let $f : K \rightarrow \mathbb{R}^m$ be a continuous function. Write $f = (f_1, \dots, f_m)$. Each f_j can be extended to a function F_j defined on \mathbb{R}^p by (46) and such that (47) is satisfied. Since for all j , $\omega(f_j, \cdot) \leq \omega(f, \cdot)$, each extension F_j is such that for all $x, y \in \mathbb{R}^p$,

$$|F_j(x) - F_j(y)| \leq \omega(f, \|x - y\|).$$

We embed the compact set K in a cube of the form $[a, b]^p$ where $a, b \in \mathbb{R}$. The extensions F_j are restricted to $[a, b]^p$ and by a change of variables, they can be defined on $[0, 1]^p$. Then, the results that will be obtained for functions defined on $[0, 1]^p$ apply to the F_j . Let \tilde{F}_j be the re-scaled version of F_j that is defined on $[0, 1]^p$. For $v, w \in [0, 1]^p$, by proposition 56, we have

$$\begin{aligned} |\tilde{F}_j(v) - \tilde{F}_j(w)| &= |F_j((\text{diam}K)v + \inf K) - F_j((\text{diam}K)w + \inf K)| \\ &\leq \omega(F_j, (\text{diam}K)\|v - w\|) \leq \omega(f_j, (\text{diam}K)\|v - w\|). \end{aligned}$$

This implies that the depth of the network is still controlled by the original function f . Indeed, we will show that the depth of the network depends on the convergence rate of the multivariate Bernstein polynomials $B_n(f_j, \cdot)$ associated to each component f_j of f . By proposition 48 and the above inequality, for all $j \in \{1, \dots, m\}$, we have

$$\|B_n(\tilde{F}_j) - \tilde{F}_j\|_\infty \leq \left(1 + \frac{p}{4}\right) \omega(\tilde{F}_j, \frac{1}{\sqrt{n}}) \leq \left(1 + \frac{p}{4}\right) \omega(f_j, \frac{\text{diam}K}{\sqrt{n}}).$$

This inequality guarantees that the extension operation does make the approximating network artificially deep.

We now turn to the proof of proposition 56. We need the following lemma that relates $\omega_c(f, \cdot)$ and $\omega(f, \cdot)$. This corresponds to (DeVore and Lorentz, 1993, Lemma 6.1).

Lemma 57 *Let $h : \mathbb{R}^p \rightarrow \mathbb{R}$ be a continuous function. Then, for all $t \geq 0$,*

$$\omega_c(h, t) \leq 2\omega(h, t).$$

Proof [Proof of Proposition 56] Since f is continuous and K is a compact set, f is bounded and thus the supremum in (46) is well-defined. Moreover, if $x \in K$, $F(x) = f(x)$ since x achieves the supremum in this case. It remains to check (47). If $x, y \notin K$, for $\epsilon > 0$, let $z \in K$ and $w \in K$ be such that

$$\begin{aligned} \frac{1}{2}(f(z) - \omega_c(f, \|x - z\|)) - \epsilon &\leq F(x) \leq \frac{1}{2}(f(z) - \omega_c(f, \|x - z\|)) + \epsilon, \\ \frac{1}{2}(f(w) - \omega(f, \|y - w\|)) - \epsilon &\leq F(y) \leq \frac{1}{2}(f(w) - \omega(f, \|y - w\|)) + \epsilon. \end{aligned}$$

Then, we have

$$\begin{aligned} F(x) - F(y) &\leq \frac{1}{2}(f(z) - \omega_c(\|x - z\|)) + \epsilon - \frac{1}{2}(f(w) - \omega_c(f, \|y - w\|)) + \epsilon \\ &\leq \frac{1}{2} \left(\omega_c(f, \|z - w\|) - \omega_c(f, \|x - z\|) + \omega_c(f, \|y - w\|) \right) \\ &\leq \frac{1}{2} \left(\omega_c(f, \|z - w\|) - ((\omega_c(f, \|x - w\|) + \omega_c(f, \|w - z\|)) + \omega_c(f, \|y - w\|)) \right) \\ &= \frac{1}{2} \left(-\omega_c(f, \|x - w\|) + \omega_c(f, \|y - w\|) \right) \\ &\leq \frac{1}{2} \left(-\omega_c(f, \|x - w\|) + \omega_c(f, \|y - x\|) + \omega_c(f, \|x - w\|) \right) \\ &= \frac{1}{2} \omega_c(f, \|y - x\|) \\ &\leq \omega(f, \|y - x\|) \end{aligned}$$

where the last inequality follows from lemma 57. By symmetry, we obtain (47) in this case. If $x \in K, y \notin K$, similar computations yield the desired result, up to an arbitrary additive $\epsilon > 0$. If $x, y \in K$, then F obviously preserves the first-order modulus of continuity $\omega(f, \cdot)$. We note that (47) also gives the continuity of the extension F . \blacksquare

B.2.2 DEPTH ESTIMATES FOR NON-POLYNOMIAL ACTIVATION FUNCTIONS

In the case where the activation function σ is non-polynomial, the proof of (Kidger and Lyons, 2020, Theorem 3.2) relies on a deep network that is constructed by "verticalizing" shallow networks and to some extent the depth of the resulting network corresponds to the sum of the widths of these shallow networks. These networks are obtained via the universal approximation theorem. Hence, to derive the estimates of Proposition 53, we should constructively prove the universal approximation theorem (Theorem 46). For this, we follow the strategy sketched in (Pinkus, 1999a, section 3).

B.2.3 DEPTH ESTIMATES FOR A SMOOTH ACTIVATION FUNCTION

We start with a lemma that provides a decomposition of a multivariate polynomial as a sum of univariate polynomials. The proof can be found in the proof of (Pinkus, 1999a, Theorem 4.1).

Lemma 58 *Let $h : \mathbb{R}^p \rightarrow \mathbb{R}$ be a polynomial of degree k . Set*

$$r := \binom{p-1+k}{k}. \quad (48)$$

Then, there exist $a^i \in \mathbb{R}^p$ and univariate polynomials p_i of degree at most k , $i = 1, \dots, r$ such that

$$h(x) = \sum_{i=1}^r p_i(\langle a^i, x \rangle).$$

We can now state and prove our first quantitative result for approximation with neural networks.

Proposition 59 *Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be a non-polynomial smooth function. Let $f : [0, 1]^p \rightarrow \mathbb{R}^m$ be a continuous function. Then, for any $\epsilon > 0$, there exists $g \in \mathcal{NN}_{p,m,p+m+2}^\sigma$ such that $\|f - g\|_\infty \leq \epsilon$. The depth of g is of order*

$$O\left(m \left(\omega^{-1}\left(f, \frac{\epsilon}{(1+\frac{p}{4})m}\right)\right)^{-2p}\right). \quad (49)$$

Proof Let $f : [0, 1]^p \rightarrow \mathbb{R}^m$ be a continuous function and write $f = (f_1, \dots, f_m)$. Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be an infinitely differentiable non-polynomial function and let $\epsilon > 0$. For $j = 1, \dots, m$, we approximate f_j by a Bernstein polynomial $B_{n_j}(f_j) : [0, 1]^p \rightarrow \mathbb{R}$

$$B_{n_j}(f_j, x) = \sum_{k_1=0}^{n_j} \cdots \sum_{k_p=0}^{n_j} f_j\left(\frac{k_1}{n_j}, \dots, \frac{k_p}{n_j}\right) p_{n_j, k_1}(x_1) \cdots p_{n_j, k_p}(x_p).$$

The n_j are chosen such that

$$\left(1 + \frac{p}{4}\right) \sum_{j=1}^m \omega\left(f_j, \frac{1}{\sqrt{n_j}}\right) \leq \epsilon. \quad (50)$$

Each $B_{n_j}(f_j, \cdot)$ is written as a sum of univariate polynomials thanks to lemma 58. For each $j \in \{1, \dots, m\}$, we have a family of vectors of $\{a_{j,1}, \dots, a_{j,r}\} \subset \mathbb{R}^p$ and univariate polynomials $\{p_{j,1}, \dots, p_{j,r}\}$ of degree at most n_j such that for all $x \in [0, 1]^p$

$$B_{n_j}(f_j, x) = \sum_{i=1}^r p_{j,i}(\langle a_{j,i}, x \rangle). \quad (51)$$

We now focus on how to approximate a given polynomial $p : \mathbb{R} \rightarrow \mathbb{R}$ by a function in $\mathcal{N}(\sigma)$, i.e by using a shallow neural network. This will tell us how to approximate the univariate polynomials in (51). For $z \in \mathbb{R}$, write $p(z)$ as $p(z) = \sum_{k=0}^n b_k z^k$. Since $\sigma \in \mathcal{C}^\infty(\mathbb{R})$ and σ is not a polynomial, there exists $\theta_0 \in \mathbb{R}$ such that, for all $k \in \mathbb{N}$, $\sigma^{(k)}(-\theta_0) \neq 0$. Moreover, we have

$$\frac{d^k}{dw^k} \sigma(wz - \theta) \Big|_{w=0, \theta=\theta_0} = z^k \sigma^{(k)}(-\theta_0).$$

Therefore, $p(z)$ can be written as

$$p(z) = \sum_{k=0}^n \frac{b_k}{\sigma^{(k)}(-\theta_0)} \frac{d^k}{dw^k} \sigma(wz - \theta) \Big|_{w=0, \theta=\theta_0}. \quad (52)$$

We now need to approximate the derivative terms. This is done via finite differences. Define

$$\Delta_h^k[\sigma](w=0, z, \theta) := \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} \sigma((k-i)hz - \theta). \quad (53)$$

Then, we have

$$\frac{d^k}{dw^k} \sigma(wz - \theta) \Big|_{w=0, \theta=\theta_0} = \frac{\Delta_h^k[\sigma](w=0, z, \theta_0)}{h^k} + O(h), \quad (54)$$

as $h \rightarrow 0$. Therefore, for all $z \in \mathbb{R}$, $p(z)$ is of the form

$$p(z) = \sum_{l=0}^N c_l \sigma(w_l z - \theta_0) + O(h), \quad (55)$$

as $h \rightarrow 0$ and where $w_l = 0$ or $w_l = (k-i)h$ for some k and i . In (55), N corresponds to the width of the one-hidden layer network that approximates p . Since there are a lot of overlaps in the evaluation of σ in (53), we simply obtain

$$N = n,$$

where we recall that n is the degree of p . This is an equality as the error term $O(h)$ in (54) can be made arbitrarily small with no extra cost regarding the width of the network.

We now return to the polynomials $B_{n_j}(f_j)$. From the previous step, each polynomial $p_{j,i}$ that appears in (51) can be computed with n_j neurons. Since there are r terms in the sum, this gives a total width of

$$W_j = rn_j = \binom{p-1+n_j}{n_j} n_j.$$

The idea behind the deep network g constructed to prove (Kidger and Lyons, 2020, Theorem 3.2) is to make shallow networks vertical. Each f_j is approximated by a shallow network g_j . Each g_j is then made vertical and these verticalised networks are then glued together to obtain the network g –see the original paper for more details. The network g outputs (g_1, \dots, g_m) and its depth is the sum of the widths of the g_j , since one neuron in g_j corresponds to one layer in g . The networks g_j can be taken to be the networks computing each $B_{n_j}(f_j)$. Hence, from the previous step, the depth of g is

$$D = \sum_{j=1}^m n_j \binom{p-1+n_j}{n_j}.$$

If we set $n_j = n$ for all j and if n is chosen accordingly to (50) with the n_j replaced by n there, then, by proposition 48,

$$\|f - g\|_\infty \leq \sum_{j=1}^m \|f_j - g_j\|_\infty \leq \sum_{j=1}^m \|f_j - B_n(f_j)\|_\infty \leq \left(1 + \frac{p}{4}\right) \sum_{j=1}^m \omega(f_j, \frac{1}{\sqrt{n}}) \leq \epsilon.$$

Therefore, the network g is such that $\|g - f\|_\infty \leq \epsilon$ and has depth $mn \binom{p-1+n}{n}$. Observe that

$$mn \binom{n+p-1}{n} = mn \frac{\prod_{k=1}^n (k+p-1)}{n!} = mn \left[\prod_{k=1}^n \frac{k+p-1}{k} \right] = mn \left[\prod_{k=1}^n 1 + \frac{p-1}{k} \right].$$

Moreover, we have

$$\exp \left(\log \left[\prod_{k=1}^n 1 + \frac{p-1}{k} \right] \right) = \exp \left(\sum_{k=1}^n \log \left(1 + \frac{p-1}{k} \right) \right) = O(n^{p-1}).$$

Therefore, the depth of g behaves like $O(mn^p)$. Thanks to (50) and since for all j , $\omega(f_j, \cdot) \leq \omega(f, \cdot)$ n can be related to $\omega(f, \cdot)$ and ϵ , showing the estimate of proposition 59. ■

B.2.4 DEPTH ESTIMATES FOR A NON-POLYNOMIAL CONTINUOUS ACTIVATION FUNCTION

To derive depth estimates in the case of a non-polynomial continuous activation function, we use convolutions to smooth the activation function thereby obtaining derivatives of all order that are used to approximate polynomials. Convolutions themselves are approximated by sums and the following lemma quantifies the error made when using such an approximation schema.

Lemma 60 *Let $\epsilon > 0$. Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be a non-polynomial continuous function and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be an infinitely differentiable function having support in an interval $[a, b]$. If $L \in \mathbb{N}$ is such that*

$$\|\phi\|_{L^1} \omega\left(\sigma, \frac{b-a}{L}\right) \leq \epsilon,$$

then there exist $c_l \in \mathbb{R}$, $y_l \in [a + (l-1)\frac{b-a}{L}, a + l\frac{b-a}{L}]$ for $l = 1, \dots, L$ such that

$$\left| (\sigma * \phi)(t) - \sum_{l=1}^L c_l \sigma(t - y_l) \right| \leq \epsilon.$$

Proof This proof is inspired by the beginning of (Cheney and Light, 2000, Chapter 24). Define the intervals

$$I_l := \left[a + (l-1)\frac{b-a}{L}, a + l\frac{b-a}{L} \right], \quad 1 \leq l \leq L.$$

They form a partition of $[a, b] = \text{supp } \phi$. Set

$$c_l = \int_{I_l} \phi(y) dy, \quad 1 \leq l \leq L. \quad (56)$$

Let y_l be in $[a + (l-1)\frac{b-a}{L}, a + l\frac{b-a}{L}]$ for $l = 1, \dots, L$. Then,

$$\begin{aligned} \left| \sigma * \phi(t) - \sum_{l=1}^L c_l \sigma(t - y_l) \right| &= \left| \int_{\text{supp } \phi} \sigma(t - y) \phi(y) dy - \sum_{l=1}^L \sigma(t - y_l) \int_{I_l} \phi(y) dy \right| \\ &\leq \sum_{l=1}^L \int_{I_l} |\sigma(t - y) - \sigma(t - y_l)| |\phi(y)| dy \\ &\leq \omega\left(\sigma, \frac{b-a}{L}\right) \left(\sum_{l=1}^L \int_{I_l} |\phi(y)| dy \right) \\ &= \|\phi\|_{L^1} \omega\left(\sigma, \frac{b-a}{L}\right) \\ &\leq \epsilon \quad \text{by assumption.} \end{aligned}$$

■

By revisiting the proof of proposition 59 and using lemma 60, we can derive the following quantitative depth estimate for deep neural networks with a non-polynomial continuous activation function.

Proposition 61 *Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be a non-polynomial continuous function which is continuously differentiable at at least one point with a nonzero derivative at that point. Let $f : [0, 1]^p \rightarrow \mathbb{R}^m$ be a continuous function. Then, for any $\epsilon > 0$, there exists $g \in \mathcal{NN}_{p, m, p+m+2}^\sigma$ such that $\|f - g\|_\infty \leq \epsilon$. The depth of g is of order*

$$O\left(m \left(\omega^{-1}\left(f, \frac{\epsilon}{2m(1 + \frac{p}{4})}\right) \right)^{-2p} \left(\omega^{-1}\left(\sigma, \frac{\epsilon}{2Bm(2^{\omega^{-1}(f, \frac{\epsilon}{2m(1 + \frac{p}{4})})^{-2+1}} - 1)}\right) \right)^{-1}\right) \quad (57)$$

for some $B > 0$ depending on f .

Proof Let $f : [0, 1]^p \rightarrow \mathbb{R}^m$ be a continuous function and let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying the assumptions of Proposition 61. To prove the result, we only need to refine the part of the proof of Proposition 59 that involves the differentiability of σ . To write the Bernstein polynomials $B_{n_j}(f_j)$ as functions in $\mathcal{N}(\sigma)$, we use the idea of the proof of (Pinkus, 1999a, Theorem 3.1).

Let $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ be such that $\sigma * \phi$ is not a polynomial. Such a function ϕ does exist, see for example the proof of (Pinkus, 1999a, Proposition 3.7). By standard properties of convolution, $\sigma * \phi$ belongs to $\mathcal{C}^\infty(\mathbb{R})$ and by considering mollifiers, we can choose a sequence $(\phi_n)_n$ such that $\sigma * \phi_n$ converges uniformly to σ on compact sets. For $t \in \mathbb{R}$,

$$\sigma * \phi(t) = \int_{-\infty}^{+\infty} \sigma(t-y)\phi(y) dy = \int_{\text{supp } \phi} \sigma(t-y)\phi(y) dy,$$

where $\text{supp } \phi$ is compact. We can assume that it is an interval of \mathbb{R} by choosing an appropriate mollifier. Moreover, for any integer $k \geq 1$:

$$\frac{d^k}{dw^k}(\sigma * \phi)(wt - \theta) = \int_{-\infty}^{+\infty} \sigma(y)t^k \phi^{(k)}(wt - \theta - y) dy.$$

Since $\sigma * \phi$ is not a polynomial, there exists $\theta_0 \in \mathbb{R}$ such that for all $k \in \mathbb{N}$, $\sigma * \phi^{(k)}(-\theta_0) \neq 0$. Then:

$$\frac{d^k}{dw^k}(\sigma * \phi)(wt - \theta) \Big|_{w=0, \theta=\theta_0} = \int_{-\infty}^{+\infty} \sigma(y)t^k \phi^{(k)}(-\theta_0 - y) dy. \quad (58)$$

Therefore, in order to express (51) as a function in $\mathcal{N}(\sigma)$, the right-hand side of (58) must be approximated as a sum. It is first approximated via finite differences:

$$\frac{d^k}{dw^k}(\sigma * \phi)(wt - \theta) \Big|_{w=0, \theta=\theta_0} = \frac{1}{h^k} \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} (\sigma * \phi)((k-i)hz - \theta_0) + O(h)$$

as $h \rightarrow 0$. Then, each term $\sigma * \phi$ is further approximated as a Riemann sum in the manner of Lemma 60. This is possible since ϕ has compact support, hence so have the $\phi^{(k)}$. For $L \in \mathbb{N}$, $t \in \mathbb{R}$, we have:

$$(\sigma * \phi)(t) \sim \sum_{l=1}^L c_l \sigma(t - y_l). \quad (59)$$

Let $p : \mathbb{R} \rightarrow \mathbb{R}$ be a polynomial of degree n , written as $p(z) = \sum_{k=0}^n b_k z^k$. Based on the above analysis, we can approximate it by an element of $\mathcal{N}(\sigma)$ and estimate the error of approximation. Observe that

$$p(z) = \sum_{k=0}^n \tilde{b}_k \frac{d^k}{dw^k}(\sigma * \phi)(wz - \theta) \Big|_{w=0, \theta=\theta_0} \quad \text{where } \tilde{b}_k = \left(\int_{-\infty}^{+\infty} \sigma(y) \phi^{(k)}(-\theta_0 - y) dy \right)^{-1} b_k. \quad (60)$$

The terms involving the derivatives of $\sigma * \phi$ are first approximated by finite differences. Then, the terms $\phi * \sigma$ that appear in the finite differences are further approximated by Riemann sums. This yields for p

$$p(z) \sim \sum_{k=0}^n \tilde{b}_k \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} \sum_{l=1}^L c_l \sigma((k-i)hz - \theta_0 - y_l). \quad (61)$$

Denote the right-hand side of (61) by $R(z)$. Notice that if we assume that z belongs to a compact set K , then we can choose ϕ such that $\|\sigma - \sigma * \phi\|_\infty$ is arbitrarily small. Here, $\|\cdot\|_\infty$ is taken on the set

$$\{(k-i)hz - \theta_0 : z \in K, i = 0, \dots, k\} = \bigcup_{i=0}^k \{(k-i)hz - \theta_0 : z \in K\}$$

which is compact since this is a finite union of compact sets. By (61), approximating $p(z)$ by $R(z)$ requires Ln evaluations of σ . To use $R(z)$ to derive depth estimates, it remains to investigate the error made when approximating $p(z)$ by $R(z)$. By Lemma 60, we have, for $z \in K$,

$$p(z) - R(z) \leq \sum_{k=0}^n \tilde{b}_k \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} (-1)^{k-i} \|\phi\|_{L^1} \omega\left(\sigma, \frac{b-a}{L}\right).$$

The right-hand term is bounded above by

$$(2^{n+1} - 1) \|\phi\|_{L^1} \omega\left(\sigma, \frac{b-a}{L}\right) \max_{0 \leq i \leq n} \tilde{b}_i.$$

Similarly, we obtain a lower bound and this yields the estimate, for $z \in K$,

$$\left| p(z) - R(z) \right| \leq (2^{n+1} - 1) \|\phi\|_{L^1} \omega\left(\sigma, \frac{b-a}{L}\right) \max_{0 \leq i \leq n} |\tilde{b}_i|. \quad (62)$$

We now come back to the setting of the proof of Proposition 61. We make the approximation

$$B_{n_j}(f_j, x) \sim \sum_{i=0}^r \sum_{k=0}^{n_j} \tilde{b}_{i,k} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \sum_{l'=1}^L c_{l'} \sigma((k-l)h\langle a_{j,i}, z \rangle - \theta_0 - y_{l'}) \quad (63)$$

for some coefficients $\tilde{b}_{i,k}$ and where the $c_{l'}$ are defined in (56). The coefficients $a_{j,i}$ come from Lemma 58. Notice that the function ϕ can be chosen such that $\|\sigma - \sigma * \phi\|_\infty$ is arbitrarily small where $\|\cdot\|_\infty$ is considered on the compact set $\cup_{i=0}^r \cup_{l=0}^k \{(k-l)h\langle a_{j,i}, z \rangle - \theta_0 : z \in [0, 1]^p\}$. Let g denote the neural network approximating f and write $g = (g_1, \dots, g_m)$. Each g_j corresponds to the approximation of $B_{n_j}(f_j)$ in (63). We have, by Proposition 48,

$$\begin{aligned} \|f - g\|_\infty &\leq \sum_{j=1}^m \|f_j - g_j\|_\infty \leq \sum_{j=1}^m \|f_j - B_{n_j}(f_j)\|_\infty + \|B_{n_j}(f_j) - g_j\|_\infty \\ &\leq \sum_{j=1}^m \left(1 + \frac{p}{4}\right) \omega\left(f_j, \frac{1}{\sqrt{n_j}}\right) + \|B_{n_j}(f_j) - g_j\|_\infty. \end{aligned}$$

Let $\epsilon > 0$. If n and L are chosen such that

$$\sum_{j=1}^m \left[\left(1 + \frac{p}{4}\right) \omega\left(f_j, \frac{1}{\sqrt{n_j}}\right) + \omega\left(\sigma, \frac{b-a}{L}\right) \|\phi\|_{L^1} (2^{n_j+1} - 1) \sum_{i=0}^r \max_{0 \leq k \leq n_j} |\tilde{b}_{i,k}| \right] \leq \epsilon, \quad (64)$$

then the network g is such that $\|f - g\|_\infty \leq \epsilon$. Its depth is the sum of the depth of the g_j . Typically, if ϕ is a mollifier, then $\|\phi\|_{L^1} = 1$. By choosing $n_j = n$ for all j and by taking each term in (64) smaller than $\epsilon/2$, we obtain the depth estimate of Proposition 61. \blacksquare

Remark 62 We see in (64) that it would be useful to have an idea of the magnitude of the coefficients $\tilde{b}_{i,k}$. However, these coefficients depend on the decomposition given by Lemma 58 of the Bernstein polynomials. To the best of our knowledge, the proof of this decomposition is not constructive and hence it does not allow us to control the magnitude of the $\tilde{b}_{i,k}$. This decomposition is actually closely related to the Waring problem which is still not totally solved. However, it can be mentioned that the value r given in (48) is an upper bound of the exact number of terms in the decomposition. Alexander and Hirschowitz (1995) show a smaller estimate for r . This could result in a decrease of the depth. As for the coefficients $b_{i,k}$, Dreesen et al. (2015) propose an algorithm to construct the decomposition of Lemma 58 which relates these coefficients to the Jacobian matrix of the multivariate polynomial being approximated. So it could be that the depth of a network approximating a function further depends on its actual smoothness, i.e not only on its first-order modulus of continuity.

B.2.5 DEPTH ESTIMATES FOR A NON-AFFINE POLYNOMIAL ACTIVATION FUNCTION

When one extra neuron on each layer is allowed By building a deep network whose construction does not rely on the universal approximation theorem, we can obtain the following quantitative result in the case of a polynomial activation function.

Proposition 63 *Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be a non-affine polynomial. Let $f : [0, 1]^p \rightarrow \mathbb{R}^m$ be a continuous function. Then, for any $\epsilon > 0$, there exists $g \in \mathcal{NN}_{p,m,p+m+3}^\sigma$ such that $\|f - g\|_\infty \leq \epsilon$. The depth of g is of order*

$$O\left(m(p+m)\left(\omega^{-1}\left(f, \frac{\epsilon}{\left(1+\frac{p}{4}\right)m}\right)\right)^{-4p-2}\right).$$

Proof Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be a non-affine polynomial. Let $f : [0, 1]^p \rightarrow \mathbb{R}^m$ be a continuous function. Verticalisation of shallow networks cannot be used anymore to construct a deep network approximating f : the universal approximation theorem rules out polynomial activation functions. To overcome this issue, an inspection of the proof of (Kidger and Lyons, 2020, Theorem 3.2) shows that it suffices to approximate each component of f by a polynomial and then count the number of multiplications that are required to evaluate each polynomial. To make the estimates meaningful, in Propositions 53 and 63, we allow an extra neuron on each layer, that is we consider feedforward neural networks with width $p + m + 3$. The case of networks with width $p + m + 2$ is treated in appendix B.2.5: Kidger and Lyons (2020) use an approximation of the inverse function $x \mapsto 1/x$ to restrict the width to $p + m + 2$, which results in an increase of the depth.

Let $\epsilon > 0$. Each component f_j of f is approximated by a Bernstein polynomial $B_n(f_j)$ of degree n with n such that

$$\left(1 + \frac{p}{4}\right) \sum_{j=1}^m \omega\left(f_j, \frac{1}{\sqrt{n}}\right) \leq \epsilon. \quad (65)$$

The network g approximating f computes these m polynomials. Its depth is given by the number of multiplications necessary to compute the monomials constituting the $B_n(f_j)$, for $j = 1, \dots, m$, see Kidger and Lyons (2020). By Definition 47, each $B_n(f_j)$ can be rewritten as

$$\begin{aligned} B_n(f_j, x) &= \sum_{k_1=0}^n \cdots \sum_{k_p=0}^n f_j \left(\frac{k_1}{n}, \dots, \frac{k_p}{n}\right) \binom{n}{k_1} \sum_{c_1=0}^{n-k_1} \binom{n-k_1}{c_1} (-1)^{n-k_1-c_1} x_1^{n-c_1} \\ &\quad \cdots \binom{n}{k_p} \sum_{c_p=0}^{n-k_p} \binom{n-k_p}{c_p} (-1)^{n-k_p-c_p} x_p^{n-c_p}. \end{aligned}$$

When k_1, \dots, k_p and c_1, \dots, c_p are fixed, we have a monomial given by

$$\gamma(x) = f_j \left(\frac{k_1}{n}, \dots, \frac{k_p}{n}\right) \prod_{j=1}^p \binom{n}{k_j} \binom{n-k_j}{c_j} (-1)^{n-k_j-c_j} x_j^{n-c_j}.$$

Computing $x_j^{n-c_j}$ requires $n - c_j - 1$ multiplications if $n - c_j \geq 1$, 0 otherwise. Let A denote the set $\{c_j : n - c_j = 0, j = 1, \dots, p\}$. Taking into account multiplications between powers of coordinates, computing γ requires

$$\left(\sum_{j=1}^p (n - c_j - 1) \mathbb{1}_{c_j < n}\right) + \sum_{j=1}^p (\mathbb{1}_{c_j < n} - \mathbb{1}_{|A|>1}) = \left(\sum_{j=1}^p n - c_j\right) - \mathbb{1}_{|A|>1} \quad (66)$$

multiplications. Above, $|A|$ denotes the cardinality of A . We now let the values of k_1, \dots, k_p and c_1, \dots, c_p vary to obtain all the monomials appearing in $B_n(f_j)$. Let M denote the total number of

monomials and let M_0 be the number of monomials where at most one coordinate x_j has a non-zero power. From (66), computing all these monomials requires

$$\left(\sum_{k_1=0}^n \cdots \sum_{k_p=0}^n \sum_{c_1=0}^{n-k_1} \cdots \sum_{c_p=0}^{n-k_p} \sum_{j=1}^p (n - c_j) \right) - M + M_0$$

multiplications. Let P denote the expression above. It can be shown that

$$P = O(n^{2p+1}),$$

see appendix B.2.6 for the details. By (Kidger and Lyons, 2020, Lemma 4.3), a multiplication can be computed by two neurons with square activation function $\rho(x) = x^2$. This implies that a neural network with square activation function must have $2 \times P$ layers to approximate $B_n(f_j)$. Therefore, a network approximating $f = (f_1, \dots, f_m)$ should have depth $2m \times P$ and width $p + m + 1$, where the value for the width is explained in Kidger and Lyons (2020). Moreover, the proof of (Kidger and Lyons, 2020, Proposition 4.11) shows that the operation of a single neuron with square activation function may be approximated by two neurons with activation function σ . Therefore, a computation neuron needs to be added to each layer. Since we also add an extra neuron to each layer, the same method can be used to approximate all the f_j , i.e use Bernstein polynomials and simply count the number of multiplications that their evaluation requires. In order to have access to the approximated values of the inputs and to the intermediate values of the $B_n(f_j)$'s, we make the network with square activation function "vertical", i.e each layer in this model becomes $p + m + 1$ layers in the network with activation function σ . Therefore, the network $g \in \mathcal{NN}_{p,m,p+m+3}^\sigma$ approximating f has depth

$$2m(p + m + 1)O(n^{2p+1}) = O(m(p + m)n^{2p+1}). \quad (67)$$

Moreover, by (65) and Proposition 48, g is such that $\|f - g\|_\infty \leq \epsilon$. By (65) and since $\omega(f_j, \cdot) \leq \omega(f, \cdot)$ for all j , n can be related to $\omega(f, \cdot)$ and ϵ , showing the estimate of Proposition 63. ■

When no extra neuron is allowed The proof of (Kidger and Lyons, 2020, Proposition 4.11) relies on an approximation of the inverse function $x \mapsto 1/x$ to avoid adding an extra neuron to each layer, i.e to approximate a function $f \in \mathcal{C}([0, 1]^p, \mathbb{R}^m)$ by a network $g \in \mathcal{NN}_{p,m,p+m+2}^\sigma$. This increases the depth of g , as shown by the following proposition.

Proposition 64 *Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be a non-affine polynomial. Let $f : [0, 1]^p \rightarrow \mathbb{R}^m$ be a continuous function. Then, for any $\epsilon > 0$, there exists $g \in \mathcal{NN}_{p,m,p+m+2}^\sigma$ such that $\|f - g\|_\infty \leq \epsilon$. For $0 < \alpha < 1$ small enough so that $[0, 1] + 1 - \alpha \subset [\frac{1}{2-\alpha}, 2 - \alpha]$, the depth of g is of order*

$$\begin{aligned} & O\left(p(p + m)\left[\omega^{-1}\left(f, \frac{\epsilon}{2m(1 + \frac{p}{4})}\right)\right]^{-4p} \log\left(\log\left(\frac{\epsilon(2 - \alpha)}{2\left[\omega^{-1}\left(f, \frac{\epsilon}{2m(1 + \frac{p}{4})}\right)\right]^{-8p}}\right)[\log(1 - \alpha)]^{-1}\right)\right) \\ & + O\left(m(p + m)\left[\omega^{-1}\left(f, \frac{\epsilon}{2m(1 + \frac{p}{4})}\right)\right]^{-4p-2}\right). \end{aligned}$$

Proof Let $f \in \mathcal{C}([0, 1]^p, \mathbb{R}^m)$ and let $\epsilon > 0$. First, for $\alpha > 0$ as in the statement of Proposition 64, we perform a change of variables $\tilde{x}_i = x_i + 1 - \alpha$, $i = 1, \dots, p$, so that the function f is defined on $[1 - \alpha, 2 - \alpha]$ and the inverse function $\tilde{x}_i \mapsto 1/\tilde{x}_i$ is well-defined for all $i = 1, \dots, p$. Each component f_j , $j = 1, \dots, m$, of f is approximated by a Bernstein polynomial $B_n(f_j)$. The polynomials $B_n(f_j)$, $j = 2, \dots, m$, are computed by a network with activation function σ satisfying the assumption of Proposition 64, as in the proof of Proposition 63. For $B_n(f_1)$, the technique used by Kidger and Lyons (2020) is first to decompose $B_n(f_1)$ into a sum of monomials. We obtain

$$M := \left(\binom{n}{2} + 1\right)(n + 1)^p$$

monomials, see appendix B.2.6. Then $B_n(f_1)$ is written as compositions of multiplications and evaluations of r_{n_a} , where r_{n_a} is defined as, for $n_a \in \mathbb{N}$,

$$r_{n_a}(x) = (2-x) \prod_{i=1}^{n_a} (1 + (1-x)^{2^i})$$

and converges to $1/x$ as $n_a \rightarrow \infty$. An inspection of the proof of (Kidger and Lyons, 2020, Lemma 4.5) shows that $r_{n_a}(\tilde{x}_i)$ for $i = 1, \dots, p$ can be computed using $3n_a$ layers. For clarity, we recall the decomposition used in Kidger and Lyons (2020)

$$\begin{aligned} \tilde{g}_1 = & \left[\prod_{k=1}^p r_{n_a}^{2M-2}(x_k)^{\theta_{1,k}} \right] \left(1 + \left[\prod_{k=1}^p r_{n_a}^{2M-3}(x_k)^{\theta_{1,k}} \right] \left[\prod_{k=1}^p r_{n_a}^{2M-4}(x_k)^{\theta_{2,k}} \right] \right. \\ & \left(1 + \left[\prod_{k=1}^p r_{n_a}^{2M-5}(x_k)^{\theta_{2,k}} \right] \left[\prod_{k=1}^p r_{n_a}^{2M-6}(x_k)^{\theta_{3,k}} \right] \right. \\ & \left(\dots \right. \\ & \left. \left(1 + \left[\prod_{k=1}^p r_{n_a}^3(x_k)^{\theta_{M-2,k}} \right] \left[\prod_{k=1}^p r_{n_a}^2(x_k)^{\theta_{M-1,k}} \right] \right. \right. \\ & \left. \left(1 + \left[\prod_{k=1}^p r_{n_a}(x_k)^{\theta_{M-1,k}} \right] \left[\prod_{k=1}^p x_k^{\theta_{M,k}} \right] \right) \right) \right) \\ & \left. \dots \right) \end{aligned} \quad (68)$$

where $r_{n_a}^\alpha$ denotes r_{n_a} composed α times. Only one neuron is available to compute (68), so this neuron has to store intermediate operations. We see that once r_{n_a} is computed for all \tilde{x}_i , with only one neuron to store values, computing the most nested set of brackets requires

$$\left(\sum_{k=1}^p \theta_{M,k} - 1 \right) + \left(\sum_{k=1}^p \theta_{M-1,k} - 1 \right) + 1 = \sum_{k=1}^p (\theta_{M,k} + \theta_{M-1,k}) - 1$$

multiplications. On the left-hand side, the $+1$ comes from the fact that once the two products have been computed, they still need to be multiplied together. Above, we have assumed that $\theta_{M,k}, \theta_{M-1,k} \geq 1$ for at least two different values of k . That is why there is an extra -1 . We then compute $r \circ r(\tilde{x}_i)$ and $r_{n_a} \circ r_{n_a} \circ r_{n_a}(\tilde{x}_i)$ for all i which requires $2 \times 3pn_a$ layers. Once $r_{n_a}^2$ and $r_{n_a}^3$ are computed, computing the second most nested set of brackets implies

$$\sum_{k=1}^p (\theta_{M-2,k} + \theta_{M-1,k}) - 1$$

multiplications, where again we have assumed that $\theta_{M-1,k}, \theta_{M-2,k} \geq 1$ for at least two distinct values of k . Assume that $\theta_{M,k} \geq 1$ for at least two different values of k . This simplifies a bit the computations and does not increase depth. Reproducing the previous reasoning for all nested brackets, we find that we need

$$(2M-2) \times 3pn_a + 2 \left(\sum_{m=1}^{M-1} \sum_{k=1}^p \theta_{M-m,k} \right) + \sum_{k=1}^p \theta_{M,k} - 2(M-M_0) + \frac{2M-2}{2} \quad (69)$$

multiplications to compute (68). As in the proof of Proposition 63, M_0 denotes the number of monomials where at most one coordinate has a non-zero power. For $m < M$, the powers $\theta_{m,k}$, $k =$

$1, \dots, p$, appear twice in (68), explaining the factor 2 in front of the second term in the sum (69). Notice that if we choose the M th polynomial to be the one where all the coordinates have power n , then the depth is a bit reduced. The last term in (69) accounts for multiplications of each bracket with the previous stored value. Since each multiplication requires two layers and that we need to make the network with approximate square activation function vertical – see proof of Proposition 63 –, the number of layers needed to approximate $B_n(f_1)$ is

$$(2M - 2)(p + m + 1) \times 3pn_a + 2(p + m + 1) \left(2 \sum_{m=1}^{M-1} \sum_{k=1}^p \theta_{M-m,k} + \sum_{k=1}^p \theta_{M,k} \right) + 2(p + m + 1)(-M + M_0 + 1). \quad (70)$$

No factor 2 appears in front of $(2M - 2) \times 3pn_a$ because $3n_a$ is already the exact number of layers necessary to compute r – see (Kidger and Lyons, 2020, Lemma 4.5). The depth of the network (70) depends on the ordering of the monomials only through the choice of the M th monomial: this M th monomial should be chosen to be the one for which the sum of powers is maximal. From the results of appendix B.2.6, we roughly have

$$\left(2 \sum_{m=1}^{M-1} \sum_{k=1}^p \theta_{M-m,k} + \sum_{k=1}^p \theta_{M,k} \right) = 2O(n^{2p+1}) = O(n^{2p+1})$$

while $M = O(n^{2p})$ and $M_0 = O(n)$. From the proof of Proposition 63, we know that for $j = 2, \dots, p$ each component g_j of g has depth $O((p + m + 1)n^{2p+1})$. Therefore, the depth of the network $g \in \mathcal{NN}_{p,m,p+m+2}^\sigma$ approximating f is of order

$$3pn_a O((p + m)n^{2p}) + O(m(p + m)n^{2p+1}). \quad (71)$$

The last term in this sum is of the same order as the order of the depth of a network $\tilde{g} \in \mathcal{NN}_{p,m,p+m+3}^\sigma$ approximating f . We thus see that the increase of depth due to removing one neuron comes from the approximation of the inverse function. To obtain a more precise estimate of the depth of g , we should investigate the rate of convergence in n_a of r_{n_a} . First, we can notice that

$$\prod_{i=0}^{n_a} (1 + x^{2^i}) = \sum_{i=0}^{2^{n_a+1}-1} x^i.$$

This can be shown by induction. For $n_a = 0$, the equality is clear. Now, assume that it holds for $n_a = k$. Then

$$\prod_{i=0}^{k+1} (1 + x^{2^i}) = (1 + x^{2^{k+1}}) \prod_{i=0}^k (1 + x^{2^i}) = (1 + x^{2^{k+1}}) \sum_{i=0}^{2^{k+1}-1} x^i = \sum_{i=0}^{2^{k+2}-1} x^i.$$

This yields, for $x \in (0, 2)$,

$$r_{n_a}(x) = \prod_{i=0}^{n_a} (1 + (1-x)^{2^i}) = \sum_{i=0}^{2^{n_a+1}-1} (1-x)^i = \frac{1 - (1-x)^{2^{n_a+1}}}{x}.$$

Therefore, the error of approximation is simply

$$\left| \prod_{i=0}^{n_a} (1 + (1-x)^{2^i}) - \frac{1}{x} \right| = \left| \frac{-(1-x)^{2^{n_a+1}}}{x} \right|.$$

In the context of the proof, x belongs to $[1 - \alpha, 2 - \alpha]$. Moreover, in (68), r_{n_a} is used to approximate $x \mapsto 1/x$ and $x \mapsto x$ alternatively. Therefore, we need to consider the error over the interval $[\frac{1}{2-\alpha}, 2 - \alpha]$, where we assume that α is taken to be less than $\frac{1}{2}(3 - \sqrt{5})$. This makes explicit the 'small enough' in the statement of Proposition 64. Then, as soon as $n_a \geq 1$, the right-hand side of the above equality reaches its maximum over $[\frac{1}{2-\alpha}, 2 - \alpha]$ at $x = 2 - \alpha$ and is therefore bounded above by

$$\eta := \frac{(1 - \alpha)^{2^{n_a+1}}}{2 - \alpha}. \quad (72)$$

In (68), the approximation r_{n_a} is composed multiple times. Thus, we need to investigate more precisely the error $\|g_1 - \tilde{g}_1\|_\infty$. By decomposing g_1 into a sum of monomials, that is $g_1 = \sum_{m=1}^M \gamma_m$, we may write, for $x \in [1 - \alpha, 2 - \alpha]^p$,

$$\begin{aligned} |\tilde{g}_1(x) - g_1(x)| &= \left| \prod_{k=1}^p r_{n_a}^{2M-2}(x_k)^{\theta_{1,k}} - \gamma_1(x) \right| \\ &+ \left| \prod_{k=1}^p r_{n_a}^{2M-2}(x_k)^{\theta_{1,k}} \prod_{k=1}^p r_{n_a}^{2M-3}(x_k)^{\theta_{1,k}} \prod_{k=1}^p r_{n_a}^{2M-4}(x_k)^{\theta_{2,k}} - \gamma_2(x) \right| \\ &+ \left| \prod_{k=1}^p r_{n_a}^{2M-2}(x_k)^{\theta_{1,k}} \prod_{k=1}^p r_{n_a}^{2M-3}(x_k)^{\theta_{1,k}} \prod_{k=1}^p r_{n_a}^{2M-4}(x_k)^{\theta_{2,k}} \prod_{k=1}^p r_{n_a}^{2M-5}(x_k)^{\theta_{2,k}} \right. \\ &\quad \left. \times \prod_{k=1}^p r_{n_a}^{2M-6}(x_k)^{\theta_{3,k}} - \gamma_3(x) \right| + \dots \end{aligned} \quad (73)$$

For the first term, we write

$$\left| \prod_{k=1}^p r_{n_a}^{2M-2}(x_k)^{\theta_{1,k}} - \gamma_1(x) \right| \leq \sum_{j=1}^{M-1} \left| \prod_{k=1}^p r_{n_a}^{2M-2j}(x_k)^{\theta_{1,k}} - \prod_{k=1}^p r_{n_a}^{2M-2(j+1)}(x_k)^{\theta_{1,k}} \right|.$$

Since $\eta < 1$ for n_a large enough, each term in the sum scales as $O(\eta)$. Therefore, the overall error scales as $O(M\eta)$. For the second term in (73), we have

$$\prod_{k=1}^p r_{n_a}^{2M-2}(x_k)^{\theta_{1,k}} \prod_{k=1}^p r_{n_a}^{2M-3}(x_k)^{\theta_{1,k}} = 1 + O(\eta).$$

Therefore, this term scales as $O(M\eta)$ too. We can repeat the argument for all the remaining terms in the sum (73). We obtain

$$\|\tilde{g}_1 - g_1\|_\infty = O(M^2\eta).$$

Now, recall that $M = O(n^{2p})$. Therefore, for the network $g = (\tilde{g}_1, g_2, \dots, g_m)$ to achieve an error of at most ϵ when approximation f , n and η should be such that

$$O(n^{4p}\eta) + \left(1 + \frac{p}{4}\right) \sum_{j=1}^m \omega(f_j, \frac{1}{\sqrt{n}}) \leq \epsilon. \quad (74)$$

Indeed, if this inequality holds, then

$$\begin{aligned}
\|f - g\|_\infty &\leq \|f_1 - \tilde{g}_1\|_\infty + \sum_{j=2}^m \|f_j - g_j\|_\infty \\
&\leq \|g_1 - \tilde{g}_1\|_\infty + \|f_1 - g_1\|_\infty + \left(1 + \frac{p}{4}\right) \sum_{j=2}^m \omega(f_j, \frac{1}{\sqrt{n}}) \\
&= O(n^{4p}\eta) + \left(1 + \frac{p}{4}\right) \sum_{j=1}^m \omega(f_j, \frac{1}{\sqrt{n}}).
\end{aligned}$$

To express the depth of g in terms of $\omega(f, \cdot)$ and ϵ , we can seek to make two terms in the sum on the left-hand side of (74) smaller than $\epsilon/2$. For the second term, this yields

$$n \geq \left(\omega^{-1}\left(f, \frac{\epsilon}{2\left(1 + \frac{p}{4}\right)m}\right)\right)^{-2}. \quad (75)$$

For the first term, we should have

$$n^{4p} \frac{(1 - \alpha)^{2^{n_a+1}}}{2 - \alpha} \leq \frac{\epsilon}{2}$$

where we used the definition (72) of η . This gives

$$n_a \geq \log\left(\frac{\log\left(\frac{\epsilon(2-\alpha)}{2n^{4p}}\right)}{\log(1-\alpha)}\right) \times \frac{1}{\log 2} - 1$$

Using (75), n_a can then be expressed in terms of ϵ and $\omega(f, \cdot)$. Finally, thanks to the estimate (71), we obtain that the depth of g is of order

$$\begin{aligned}
&O\left(p(p+m)\left[\omega^{-1}\left(f, \frac{\epsilon}{2m\left(1 + \frac{p}{4}\right)}\right)\right]^{-4p} \log\left(\log\left(\frac{\epsilon(2-\alpha)}{2\left[\omega^{-1}\left(f, \frac{\epsilon}{2m\left(1 + \frac{p}{4}\right)}\right)\right]^{-8p}}\right)[\log(1-\alpha)]^{-1}\right)\right) \\
&+ O\left(m(p+m)\left[\omega^{-1}\left(f, \frac{\epsilon}{2m\left(1 + \frac{p}{4}\right)}\right)\right]^{-4p-2}\right).
\end{aligned}$$

■

B.2.6 ESTIMATES OF THE NUMBER OF MULTIPLICATIONS

We provide a proof for the estimate of P used in the proof of Proposition 63. The first term appearing in P is

$$\sum_{k_1=0}^n \cdots \sum_{k_p=0}^n \sum_{c_1=0}^{n-k_1} \cdots \sum_{c_p=0}^{n-k_p} \sum_{j=1}^p (n - c_j).$$

The sum is split into two terms:

$$\begin{aligned}
P_1 &= \sum_{k_1=0}^n \cdots \sum_{k_p=0}^n \sum_{c_1=0}^{n-k_1} \cdots \sum_{c_p=0}^{n-k_p} np, \\
P_2 &= \sum_{k_1=0}^n \cdots \sum_{k_p=0}^n \sum_{c_1=0}^{n-k_1} \cdots \sum_{c_p=0}^{n-k_p} \sum_{j=1}^p c_j.
\end{aligned}$$

For P_1 , we have:

$$\begin{aligned}
 P_1 &= \sum_{k_1=0}^n \cdots \sum_{k_p=0}^n np \prod_{j=1}^p (n - k_j + 1) \\
 &= np \sum_{k_1=0}^n \cdots \sum_{k_{p-1}=0}^n \left[\prod_{j=1}^{p-1} (n - k_j + 1) \right] \sum_{k_p=0}^n (n - k_p + 1) \\
 &= np \sum_{k_1=0}^n \cdots \sum_{k_{p-1}=0}^n \left[\prod_{j=1}^{p-1} (n - k_j + 1) \right] \left(\frac{n}{2} + 1 \right) (n + 1) \\
 &= np \left(\frac{n}{2} + 1 \right) (n + 1) \sum_{k_1=0}^n \cdots \sum_{k_{p-2}=0}^n \left[\prod_{j=1}^{p-2} (n - k_j + 1) \right] \sum_{k_{p-1}=0}^n (n - k_{p-1} + 1) \\
 &= np \left(\left(\frac{n}{2} + 1 \right) (n + 1) \right)^2 \sum_{k_1=0}^n \cdots \sum_{k_{p-2}=0}^n \left[\prod_{j=1}^{p-2} (n - k_j + 1) \right] \\
 &= np \left(\left(\frac{n}{2} + 1 \right) (n + 1) \right)^p = O(n^{2p+1}).
 \end{aligned}$$

For P_2 , we have:

$$P_2 = \sum_{k_1=0}^n \cdots \sum_{k_p=0}^n \sum_{c_1=0}^{n-k_1} \cdots \sum_{c_p=0}^{n-k_p} \sum_{j=1}^{p-1} c_j + \sum_{k_1=0}^n \cdots \sum_{k_p=0}^n \sum_{c_1=0}^{n-k_1} \cdots \sum_{c_p=0}^{n-k_p} c_p.$$

Let P_3 denote the second term of this sum. We have:

$$\begin{aligned}
 P_3 &= \sum_{k_1=0}^n \cdots \sum_{k_p=0}^n \sum_{c_1=0}^{n-k_1} \cdots \sum_{c_{p-1}=0}^{n-k_{p-1}} \frac{(n - k_p)(n - k_p + 1)}{2} \\
 &= \sum_{k_1=0}^n \cdots \sum_{k_p=0}^n \frac{(n - k_p)(n - k_p + 1)}{2} \left[\prod_{j=1}^{p-1} (n - k_j + 1) \right] \\
 &= \sum_{k_1=0}^n \cdots \sum_{k_p=0}^n \frac{n - k_p}{2} \left[\prod_{j=1}^p (n - k_j + 1) \right] \\
 &= \frac{n}{2} \sum_{k_1=0}^n \cdots \sum_{k_p=0}^n \left[\prod_{j=1}^p (n - k_j + 1) \right] \\
 &\quad - \frac{1}{2} \sum_{k_1=0}^n \cdots \sum_{k_p=0}^n k_p \left[\prod_{j=1}^p (n - k_j + 1) \right] \\
 &= \frac{n}{2} \left(\left(\frac{n}{2} + 1 \right) (n + 1) \right)^p \\
 &\quad - \frac{1}{2} \sum_{k_1=0}^n \cdots \sum_{k_p=0}^n k_p \left[\prod_{j=1}^p (n - k_j + 1) \right].
 \end{aligned}$$

The second term roughly behaves like $O(n^{2p-1})$: this can be seen by taking $k_p = n$ and $k_j = 0$ for all $j = 1, \dots, p-1$. Thus P_3 is of order $O(n^{2p+1})$. We apply the same decomposition for the first term of P_2 and the computations are similar to those for P_3 . Therefore, we obtain

$$P_2 = O(n^{2p+1}).$$

Now, to get the rough behavior of P , we still have to take into account the term $M - M_0$. On one hand, the total number M of monomials is equal to

$$\begin{aligned} M &= \sum_{k_1=0}^n \cdots \sum_{k_p=0}^n \sum_{c_1=0}^{n-k_1} \cdots \sum_{c_p=0}^{n-k_p} 1 \\ &= \sum_{k_1=0}^n \cdots \sum_{k_p=0}^n \left[\prod_{j=1}^p (n - k_j + 1) \right] \\ &= \left(\left(\frac{n}{2} + 1 \right) (n + 1) \right)^p \\ &= O(n^{2p}). \end{aligned}$$

On the other hand, there are np monomials where only one coordinate has a non zero power and 1 monomial where all the coordinates have 0 power. Therefore, $M - M_0$ behaves like $O(n^{2p})$. This yields for P

$$P = O(n^{2p+1}).$$

B.3 Proofs of GDN Approximation Results

B.3.1 PROOF OF THEOREM 9

Proof [Proof of Theorem 9] We take $\mathcal{F} = \mathcal{NN}_{m,n,m+n+2}^\sigma$. Let $f \in C(\mathcal{X}, \mathcal{Y})$. Let $x \in \mathcal{X}$. By Lemma 52, on $\overline{B_{\mathcal{X}}(x, \delta)}$, f can be represented by

$$f = \text{Exp}_{\mathcal{Y}, f(x)} \circ \tilde{f} \circ \text{Exp}_{\mathcal{X}, x}^{-1}$$

where $\tilde{f} \in \mathcal{C}(\mathbb{R}^m, \mathbb{R}^n)$. Since $\text{Exp}_{\mathcal{X}, x}^{-1}$ is continuous, $K := \text{Exp}_{\mathcal{X}, x}^{-1}(\overline{B_{\mathcal{X}}(x, \delta)}) \subset \mathbb{R}^m$ is compact. We can consider \tilde{f} on K and then approximate it with a neural network $g \in \mathcal{NN}_{m,n,m+n+2}$. By choosing g such that $\|\tilde{f} - g\|_\infty \leq L_{\mathcal{Y}, f(x)}^{-1} \epsilon$ on K

$$\|f - \text{Exp}_{\mathcal{Y}, f(x)} \circ g \circ \text{Exp}_{\mathcal{X}, x}^{-1}\| \leq L_{\mathcal{Y}, f(x)} \|\tilde{f} \circ \text{Exp}_{\mathcal{X}, x}^{-1} - g \circ \text{Exp}_{\mathcal{X}, x}^{-1}\| \leq \epsilon.$$

The depth of g is related to \tilde{f} via its inverse modulus of continuity. By Lemma 52, it is given by

$$\omega(\tilde{f}, \epsilon) = L_{\mathcal{Y}, f(x)}^{-1} \omega\left(f, L_{\mathcal{X}, x}^{-1} \epsilon\right)$$

This equality is a consequence of the right continuity of the generalized inverse of modulus of continuity as shown in Embrechts and Hofert (2013). The result now follows from Proposition 53 with $K \triangleq \overline{B_{\mathcal{X}}(x, \delta)}$. Note, in the theorem's statement we have set $\kappa_1 \triangleq L_{\mathcal{X}, x}$ and $\kappa_2 \triangleq L_{\mathcal{Y}, f(x)}^{-1}$. ■

B.3.2 PROOF OF COROLLARY 12

Proof [Proof of Corollary 12] By definition, \mathcal{X} and \mathcal{Y} have everywhere non-positive sectional curvature $K \leq 0$. Hence, $k_{\mathcal{X}}^* = k_{\mathcal{Y}}^* = \infty$ for each $x \in \mathcal{X}$ and each $y \in \mathcal{Y}$. In particular, this is the case for $y = f(x)$ for any $f \in \overline{C}(\mathcal{X}, \mathcal{Y})$. The result now following from the left-hand side of the estimate in (4) within Theorem 9. ■

B.4 Proof of Results of Uncursed Approximation Results of Section 3.3

Proof [Proof of Proposition 17] Fix $f \in \bar{C}(\mathcal{X}, \mathcal{Y})$, $x \in \mathcal{X}$, $0 \leq \eta < 1$, and let $\emptyset \neq \mathbb{X} \subseteq \mathcal{X}$ such that condition (6) holds. Since $\text{Exp}_{\mathcal{Y}, f(x)}$ and $\text{Exp}_{\mathcal{X}, x}^{-1}$ are smooth and since we have assumed (6) then, for each $i = 1, \dots, D$, each of the functions is well-defined and belongs to $C^{nd,1}(\overline{B_{\mathbb{R}^p}(0, \eta \text{inj}_{\mathcal{X}}(x))}, \mathbb{R}^m)$:

$$\tilde{f}_i \triangleq \pi_i \circ \text{Exp}_{\mathcal{Y}, f(x)} \circ f \circ \text{Exp}_{\mathcal{X}, x}^{-1}.$$

For each $c = 1, \dots, C^*$ and $i = 1, \dots, m$ define the data:

$$\tilde{x}_{c,i} \triangleq \text{Exp}_{\mathcal{X}, x}^{-1}(x_c)_i.$$

Observe that, if $C^* > \#C$ then C^* is precisely the number of multi-indices $(\beta_1, \dots, \beta_p)$ satisfying $\sum_{i=1}^p \beta_i \leq nd - 1$. Otherwise, it is impossible to pick more $\#C$ distinct points in C (in particular we cannot pick $(C+1)^{2^C}$ distinct points). In either case, C^* is the quantity $k^\#$ defined on (Fefferman, 2005, page 576). Hence the conditions hold by discussion on (Fefferman, 2005, page 510) following (Fefferman, 2005, Theorem A)'s statement.

For the last statement, we only need to show that condition (6) is always satisfied if \mathcal{X} and \mathcal{Y} are Cartan-Hadamard manifolds. This is true because if \mathcal{X} and \mathcal{Y} are both Cartan-Hadamard manifolds, then the Cartan-Hadamard Theorem ((Jost, 2017, Corollary 6.9.1)) implies that $\text{inj}_{\mathcal{X}}(x) = \infty = \text{inj}_{\mathcal{Y}}(y)$ for every $x \in \mathcal{X}$ and every $y \in \mathcal{Y}$; thus (6). \blacksquare

Proof [Proof of Corollary 18] By (Hirsch, 1994, Theorem 2.4), the set of smooth functions from \mathcal{X} to \mathcal{Y} (denoted by $C^\infty(\mathcal{X}, \mathcal{Y})$) is dense in $\bar{C}(\mathcal{X}, \mathcal{Y})$. Therefore, it is enough to show the result holds when $f \in C^\infty(\mathcal{X}, \mathcal{Y})$. Fix an atlas $(\phi_\alpha, U_\alpha)_{\alpha \in A}$ of \mathcal{X} and an atlas $(\rho_\gamma, V_\gamma)_{\gamma \in \Gamma}$ of \mathcal{Y} . Fix any ϕ_α and ρ_γ such that the composition:

$$\phi_\alpha \circ f \circ \rho_\gamma^{-1},$$

is well-defined. Since ϕ_α and ρ_γ are smooth and f is smooth by hypothesis, then we have $\phi_\alpha \circ f \circ \rho_\gamma^{-1} \in C^\infty(K, \mathbb{R}^m)$ for any non-empty compact subset $K \subseteq \phi_\alpha(U_\alpha)$. By the Mean-Valued Theorem, every such smooth function has Lipschitz k^{th} -order partial derivatives locally on any compact subset $\emptyset \neq K \subseteq \phi_\alpha(U_\alpha)$ in any smooth chart (ϕ_α, U_α) on \mathcal{X} . Therefore, $C^\infty(\mathcal{X}, \mathcal{Y}) \subseteq C_{\text{loc}}^{k,1}(\mathcal{X}, \mathcal{Y})$ for any $k \in \mathbb{N}_+$; in particular this is the case for $k = d$. The conclusion now follows from Proposition 17. \blacksquare

Proof [Proof of Proposition 19] Since $\mathcal{X} \subset B_{\mathcal{X}}(x, \eta \mathcal{U}_f(x))$, then Lemma 51 implies that $\tilde{\mathbb{X}} \triangleq \{\text{Exp}_{\mathcal{X}, x}^{-1}(z)\}_{z \in \mathbb{X}}$ is well-defined. Moreover, Lemma 51 implies that (for each $c = 1, \dots, \#\mathbb{X}$) the following Lagrange-type polynomial, of degree exactly $\#\mathbb{X} - 1$, is well-defined:

$$p_c(z) \triangleq \sum_{\tilde{x} \in \tilde{\mathbb{X}}} \frac{\prod_{y \neq \tilde{x}} (z - y)}{\prod_{y \neq \tilde{x}} (\tilde{x} - y)} \text{Exp}_{\mathcal{Y}, f(x)}^{-1} \circ f \circ \text{Exp}_{\mathcal{X}, x}(\tilde{x}). \quad (76)$$

Since \mathbb{X} is finite, then $\overline{B_{\mathbb{R}^p}(0, \text{diam}(\tilde{\mathbb{X}}))}$ is closed and bounded. Thus by the Heine-Borel Theorem it is compact in \mathbb{R}^p . Hence, the quantity:

$$M \triangleq 2 \text{diam}(\tilde{\mathbb{X}}) \max_{c=1, \dots, \#\mathbb{X}, \tilde{x} \in \tilde{\mathbb{X}}, \|z\| \leq \|\tilde{x}\|} \|p_c(z)\| + \sum_{|\beta| \leq p \#\mathbb{X}} |\partial^\beta p_c(z)|,$$

must be finite. By construction, M satisfies Definition 16 (ii) and (iii). Observe that, Definition (i) holds by construction of the polynomials $p_1, \dots, p_{\#\mathbb{X}}$. Since each p_c is of degree exactly $\#\mathbb{X} - 1$ then the condition that $p+1 < \#\mathbb{X}$ and p divides $\#\mathbb{X} - 1$ implies that $\frac{\#\mathbb{X}}{p} \in \mathbb{N}_+$; thus, \mathbb{X} is $\frac{\#\mathbb{X}-1}{p}$ -efficient. \blacksquare

B.4.1 PROOF OF THEOREM 20

In the following proof, denote use $\pi_i : \mathbb{R}^m \ni x \mapsto x_i \in \mathbb{R}$ to denote each of the canonical coordinate projections (for $i = 1, \dots, m$). We also use e_1, \dots, e_m to denote the standard orthonormal basis on \mathbb{R}^m .

Proof [Proof of Theorem 20] Since \mathbb{X} is normalizable, then there must exist some $\eta \in [0, 1)$ and some $x \in \mathcal{X}$ such that:

$$\mathbb{X} \subseteq B_{\mathcal{X}}(x, \eta \mathcal{U}_f(x)), \quad (77)$$

$$\tilde{\mathbb{X}} \triangleq \text{Exp}_{\mathcal{X}, x}^{-1}[\mathbb{X}] \subseteq [0, 1]^p. \quad (78)$$

By Lemma 52, there is a $\tilde{f} : \mathbb{R}^m \rightarrow \mathbb{R}^p$ satisfying:

$$\text{Exp}_{\mathcal{Y}, f(x)}^{-1} \circ f \circ \text{Exp}_{\mathcal{X}, x}|_{\tilde{\mathbb{X}}} = \tilde{f}|_{\tilde{\mathbb{X}}} = \sum_{i=1}^m (\pi_i \circ \tilde{f}|_{\tilde{\mathbb{X}}}) e_i. \quad (79)$$

Since f was assumed to be n -efficient then, for each $i = 1, \dots, m$, the sets: $\{(\pi_i \circ \tilde{f}(z), z) : z \in \tilde{\mathbb{X}}\}$ satisfy the conditions of the Whitney-type Extension Theorem (as formulated in (Fefferman, 2005, Theorem A)). Therefore, there exist: $F_1, \dots, F_m \in C^{nd,1}(\mathbb{R}^p, \mathbb{R})$ satisfying:

$$F_i|_{\tilde{\mathbb{X}}} = \pi_i \circ \tilde{f}|_{\tilde{\mathbb{X}}}; \quad i = 1, \dots, m. \quad (80)$$

Since $F_i \in C^{nd,1}(\mathbb{R}^p, \mathbb{R})$ then (Yarotsky and Zhevnerchuk, 2020, Theorem 3.3) implies that, for every $\epsilon > 0$, there are $\hat{g}_i \in \mathcal{NN}_{p,1}^{\max\{0,\cdot\}}$ satisfying the uniform approximation estimate:

$$\max_{i=1, \dots, m} \sup_{x \in [0,1]^p} |\hat{g}_i(x) - F_i(x)| < \epsilon. \quad (81)$$

Therefore, incorporating the identities (79) and (80) into the estimate (81) we may derive the following approximation bound:

$$\sup_{x \in \mathbb{X}} d_{\mathcal{Y}} \left(\text{Exp}_{\mathcal{Y}, f(x)} \left(\sum_{i=1}^m \hat{g}_i \circ \text{Exp}_{\mathcal{X}, x}^{-1}(x) e_i \right), f(x) \right) \quad (82)$$

$$= \sup_{x \in \mathbb{X}} d_{\mathcal{Y}} \left(\text{Exp}_{\mathcal{Y}, f(x)} \left(\sum_{i=1}^m \hat{g}_i \circ \text{Exp}_{\mathcal{X}, x}^{-1}(x) e_i \right), \text{Exp}_{\mathcal{Y}, f(x)} \circ \tilde{f} \circ \text{Exp}_{\mathcal{X}, x}^{-1}(x) \right) \quad (83)$$

$$\leq L_{\mathcal{Y}, f(x)}^{-1} \sup_{x \in \mathbb{X}} \left\| \sum_{i=1}^m \hat{g}_i \circ \text{Exp}_{\mathcal{X}, x}^{-1}(x) e_i - \tilde{f} \circ \text{Exp}_{\mathcal{X}, x}^{-1}(x) \right\| \quad (84)$$

$$\leq L_{\mathcal{Y}, f(x)}^{-1} \sup_{z \in \tilde{\mathbb{X}}} \left\| \sum_{i=1}^m \hat{g}_i(z) e_i - \tilde{f}(z) \right\|$$

$$\leq L_{\mathcal{Y}, f(x)}^{-1} \sup_{z \in [0,1]^p} \left\| \sum_{i=1}^m \hat{g}_i(z) e_i - \tilde{f}(z) \right\|$$

$$\leq L_{\mathcal{Y}, f(x)}^{-1} m^{\frac{1}{2}} \sup_{z \in [0,1]^p} |\hat{g}_i(z) - \pi_i \circ \tilde{f}(z)|$$

$$\leq L_{\mathcal{Y}, f(x)}^{-1} m^{\frac{1}{2}} \epsilon; \quad (85)$$

where, the pass from (83) to (84) was implied by the Lipschitz estimate in Lemma 52. In particular, (Yarotsky, 2017, Theorem 3.4) implies that, for $i = 1, \dots, m$, $\hat{g}_i \in \mathcal{NN}_{p,1:2p+1}^{\sigma}$ each with $\mathcal{O}(\epsilon^{-\frac{2p}{3(np+1)}})$ trainable weights each of depth of order $\mathcal{O}(\epsilon^{\frac{2p}{3(np+1)} - \frac{p}{np+1}})$ (since each F_i belonged to $C^{nd,1}(\mathbb{R}^d, \mathbb{R})$).

It remains to show that $\sum_{i=1}^m \hat{g}_i e_i$ can be implemented by DNN in $\mathcal{NN}_{p,m}^\sigma$. Indeed, since σ was non-affine and piecewise linear, then $B \in \mathbb{N}_+$ (an the notation at the start of Section 3.3.3), breaks then (Yarotsky, 2017, Proposition 1 (b)) implies that there are $g_1, \dots, g_m \in \mathcal{NN}_{p,1:W_i}^\sigma$ such that for every $x \in [0, 1]^p$ and each $i = 1, \dots, m$ we have:

$$\hat{g}_i(x) = g_i(x). \quad (86)$$

Moreover, each g_i has the same depth as \hat{g}_i ; i.e., each g_i has depth $\mathcal{O}\left(\epsilon^{\frac{2p}{3(np+1)} - \frac{p}{np+1}}\right)$, each g_i has width at-most $2(2p+1)$, and each g_i has width at-most $4 \times$ that of the corresponding \hat{g}_i ; i.e., each g_i has $\mathcal{O}\left(\epsilon^{-\frac{2p}{3(np+1)}}\right)$ trainable weights.

Next, we observe that by (Yarotsky, 2017, Proposition 1 (b)) and the discussion on (Cheridito et al., 2021, page 3, following Definition 4) $\mathcal{NN}_{1,1}^\sigma$ has the 8-identity requirement (defined in (Cheridito et al., 2021, Definition 4)). Therefore, (Cheridito et al., 2021, Proposition 5) applies; whence, there is a network $\hat{g} \in \mathcal{NN}_{p,m}^\sigma$ "parallelizing the g_1, \dots, g_m i.e. g satisfies:

$$g = \sum_{i=1}^m g_i e_i. \quad (87)$$

Moreover, \hat{g} has depth equal to m plus the sum of the depths of $\hat{g}_1, \dots, \hat{g}_m$; i.e. it has depth of order $\mathcal{O}\left(m + m\epsilon^{\frac{2p}{3(np+1)} - \frac{p}{np+1}}\right)$, it has width equal to the sum of the widths of the $\hat{g}_1, \dots, \hat{g}_m$ plus $m8$; i.e. \hat{g} has width at-most $m(4p+10)$ but no less than m , and \hat{g} has at-most $\left(\frac{11}{4}8^2m^2m^2 - 1\right) \sum_{i=1}^m p_i$ trainable parameters (where p_i is the number of trainable parameters in \hat{g}_i); thus, \hat{g} has at-most $\mathcal{O}\left(m(m^2 - 1)\epsilon^{-\frac{2p}{3(np+1)}}\right)$ trainable parameters. Combining the identities (87) and (86) and plugging them into (82)- 85 we obtain the desired estimate:

$$\begin{aligned} & \sup_{x \in \mathcal{X}} d_{\mathcal{Y}} \left(\text{Exp}_{\mathcal{Y}, f(x)} \circ g \circ \text{Exp}_{\mathcal{X}, x}^{-1}(x), f(x) \right) \\ &= \sup_{x \in \mathcal{X}} d_{\mathcal{Y}} \left(\text{Exp}_{\mathcal{Y}, f(x)} \left(\sum_{i=1}^m \hat{g}_i \circ \text{Exp}_{\mathcal{X}, x}^{-1}(x) e_i \right), f(x) \right) \\ &\leq L_{\mathcal{Y}, f(x)}^{-1} m^{\frac{1}{2}} \epsilon. \end{aligned}$$

This concludes our proof. ■

B.5 Proofs of The Results for GDL models of Section 4

Proof [Proof of Theorem 31] Together, Assumption 4, Assumption 5, and Assumption 6 implies that \mathcal{X}_0 and \mathcal{F} are both simply connected; hence, (Burago et al., 2001, Proposition 3.4.15) applies; whence, the projection map (16) is a covering map.

Since \mathcal{X} satisfies Assumption 5 (i) (with \mathcal{X} in place of \mathcal{F}) and since \mathcal{X} is also a manifold, then it is path-connected and locally path-connected; thus (Spanier, 1995a, Chapter 2, Section 2, Theorem 5) applies and therefore there exists some $\tilde{f} \in C(\mathcal{X}, \mathcal{F})$ satisfying the following "lifting property":

$$f(x) = \rho_1 \circ \tilde{f}(x) \quad (\forall x \in \mathcal{X}). \quad (88)$$

Applying Theorem 9, we conclude that there exists some GDN \hat{f} (with depth order recorded in Table 1) of width at-most $d + m + 2$ satisfying the approximation bound:

$$\max_{x \in \mathcal{X}} d_{\mathcal{F}}(\tilde{f}(x), \hat{f}(x)) < \epsilon. \quad (89)$$

By (Burago et al., 2001, Lemma 3.3.6), $\bar{d}_{\mathcal{F}}$ coincides with the quotient metric on \mathcal{F} from (Burago et al., 2001, Definition 3.1.12) and the discussion at the bottom of (Burago et al., 2001, page 62) we conclude that:

$$\bar{d}_{\mathcal{F}}([z_1], [z_2]) \leq d_{\mathcal{F}}(\tilde{z}_1, \tilde{z}_2), \quad (90)$$

for every $z_1, z_2 \in \mathcal{F}$ and every $\tilde{z}_1 \in \rho_1^{-1}[z_1]$ and every $\tilde{z}_2 \in \rho_1^{-1}[z_2]$. In particular, for each $x \in \mathbb{X}$, (90) holds for $[z_1] = [\tilde{f}(x)]$ and $[z_2] = [\hat{f}(x)]$. Now, since ρ_1 is a surjection, then by construction of ρ_1 we have:

$$[\tilde{f}(x)] = \rho_1(\tilde{f}(x)) \text{ and } [\hat{f}(x)] = \rho_1(\hat{f}(x)). \quad (91)$$

Plugging (91) into (90) and incorporating the result into (89) yields the estimate:

$$\sup_{x \in \mathbb{X}} \bar{d}_{\mathcal{F}}(\rho_1(\tilde{f}(x)), \rho_1(\hat{f}(x))) \leq d_{\mathcal{F}}(\tilde{z}_1, \tilde{z}_2) < \epsilon. \quad (92)$$

By (88) we may replace the quantity $\rho_1(\tilde{f}(x))$ in (92) with $f(x)$; therefore, we find that:

$$\sup_{x \in \mathbb{X}} \bar{d}_{\mathcal{F}}(f(x), \rho_1(\hat{f}(x))) \leq d_{\mathcal{F}}(\tilde{z}_1, \tilde{z}_2) < \epsilon. \quad (93)$$

This concludes the proof. \blacksquare

Proof [Proof of Corollary 33] By (Munkres, 2000, Theorem 19.6) $f \in C(\mathcal{X}, \prod_{i=1}^I \mathcal{Y}_i)$ if and only if, for $k = 1, \dots, I$, there exist $f_i \in C(\mathcal{X}, \mathcal{Y}_i)$ such that:

$$f = (f_1, \dots, f_I). \quad (94)$$

Since each \mathcal{Y}_i and \mathcal{F}_i satisfy Assumptions 4 and 5 then, for every $0 < \delta_i < \mathcal{U}_{f_i}(x)$ there exist $g_i \in \mathcal{NN}_{p, m_i}^{\sigma}$ (with width $p + m_i + 2$ depth recorded in Table 1 with m_i in place of m) such that:

$$\sup_{x \in \mathbb{X}} d_{\mathcal{Y}_i}(f_i(x), \hat{f}_i(x)) < \epsilon; \quad (95)$$

where each of the \hat{f}_i are as in (19). Thus, together, (94) and (95) imply the estimate:

$$\begin{aligned} \sup_{x \in \mathbb{X}} d_{\prod_{i=1}^I \mathcal{Y}_i}(f(x), (\hat{f}_1(x), \dots, \hat{f}_I(x))) &= \sup_{x \in \mathbb{X}} d_{\prod_{i=1}^I \mathcal{Y}_i}((f_1(x), \dots, f_I(x)), (\hat{f}_1(x), \dots, \hat{f}_I(x))) \\ &= \sup_{x \in \mathbb{X}} \max_{k=1, \dots, I} d_{\mathcal{Y}_k}(f_k(x), \hat{f}_k(x)) < \epsilon. \end{aligned}$$

Lastly, consider the "efficient case"; i.e. when $\rho_i = 1_{\mathcal{F}_i}$ and \mathbb{X} is f_i -normalized and n -efficient for f_i , for each $i = 1, \dots, I$. Then, the \hat{f}_i implementing the estimates (95) are instead given by Theorem 20. \blacksquare

Proof [Proof of Proposition 35] By Assumption 8 (i), $\text{Im}(\rho) = R^{-1} \left[\prod_{i=1}^I \mathcal{Y}_i \right]$, R is continuous, and therefore $\text{Im}(\rho)$ is an open subset of \mathcal{Y} . To show (i), it only remains to show that $\text{Im}(\rho)$ is dense in \mathcal{Y} . Let $y \in \mathcal{Y}$ and define the sequence $(y_n)_{n \in \mathbb{N}}$ by $y_n \triangleq H_{1-n^{-1}}(y)$ and note that by Assumption 8 (ii) point (a), $\{y_n\}_{n \in \mathbb{N}} \in \text{Im}(\rho)$. By Assumption 8 (ii) point (b), we have that $\lim_{n \rightarrow \infty} d_{\mathcal{Y}}(y_n, y) = 0$; thus, $\text{Im}(\rho)$ is dense in \mathcal{Y} .

Suppose that \mathcal{Y} is a topological manifold with boundary whose topology is induced by the metric $d_{\mathcal{Y}}$; i.e. \mathcal{Y} is a metrizable topological manifold with boundary (see Brown (1962) for example). Since $\text{Im}(\rho)$ is an open subset of \mathcal{Y} , then $Z := \mathcal{Y} - \text{Im}(\rho)$ is a closed subset of \mathcal{Y} with the property that there exists a homotopy $h : [0, 1] \times \mathcal{Y} \rightarrow \mathcal{Y}$ satisfying $h(0, \cdot) = 1_{\mathcal{Y}}$ and for which $h(t, \mathcal{Y}) \subseteq \mathcal{Y} - Z$ for all $t \in (0, 1]$; namely $h(y, t) := H_{1-t}(y)$. Thus, Z is a \mathcal{Z} -set (as defined at the beginning of

(Guilbault and Moran, 2019, Section 3)). Hence, (Guilbault and Moran, 2019, Example 3) implies that Z is contained in \mathcal{Y} 's boundary; thus, (ii) holds. \blacksquare

Proof [Proof of Theorem 37] Fix $\epsilon > 0$. By Assumption 8, there exist a $t_{\frac{\epsilon}{2}} \in [0, 1]$ satisfying:

$$\sup_{y \in \mathcal{Y}} d_{\mathcal{Y}}(y, H_{t_{\frac{\epsilon}{2}}}(y)) < 2^{-1}\epsilon \text{ and } H_{t_{\frac{\epsilon}{2}}}(\mathcal{Y}) \subseteq \text{Im}(\rho). \quad (96)$$

Since \mathcal{X} is (non-empty) and compact and since ϕ is continuous, then (Munkres, 2000, Theorem 26.5) implies that $\phi(\mathcal{X}) \subseteq \mathcal{X}_0$ is (non-empty) compact.

By Assumption 8 (i), there is a continuous $R \in C(\text{Im}(\rho), \prod_{k=1}^K \mathcal{Y}_k)$ which is a right-inverse for ρ on $\text{Im}(\rho)$. Since H is continuous for the product topology, then (Munkres, 2000, Theorem 18.4) implies that $H_{t_{\frac{\epsilon}{2}}} \in C(\mathcal{Y}, \mathcal{Y})$. Since ϕ is continuous and \mathcal{X} is compact then the Closed Map Lemma (see (Munkres, 2000, Exercise 26.6)) implies that ϕ is injective and proper (see nLab authors (2021)); thus, ϕ^{-1} exists and belongs to $C(\phi(\mathcal{X}), \mathcal{X})$. Therefore, the coincidence between the uniform convergence on compact sets topology and the compact-open topology (see (Munkres, 2000, Theorem 46.8) and (Munkres, 2000, Exercise 46.7)) implies that $R \circ H_{t_{\frac{\epsilon}{2}}} \circ f \circ \phi^{-1} \in C(\phi(\mathcal{X}), \prod_{k=1}^K \mathcal{Y}_k)$.

Since K is a non-empty compact subsets of

$$B_{\mathcal{X}}(x^*, \tilde{\omega}_{\phi}^{-1}(\eta \min_{k=1, \dots, K} \mathcal{U}_{[R \circ H_{t_{\frac{\epsilon}{2}}} \circ f \circ \phi^{-1}]_k}(x^*))).$$

Thus, the continuity of $\tilde{\omega}_{\phi}$ and (Ebrechts and Hofert, 2013, Proposition 1 (4)) implies that:

$$\phi(K) \subset B_{\mathcal{X}_0}(\phi(x^*), \eta \min_{k=1, \dots, K} \mathcal{U}_{[R \circ H_{t_{\frac{\epsilon}{2}}} \circ f \circ \phi^{-1}]_k}(x^*)).$$

Since, moreover, σ satisfies Assumption 1 then, the conditions of Corollary 33 are met. Therefore, for each $k = 1, \dots, K$, there exist $g \in \mathcal{NN}_{p, m_k}^{\sigma}$ such that $\hat{f}' \triangleq (\rho_1 \circ \text{Exp}_{\mathcal{F}_1, y_1} \circ f \circ \text{Exp}_{\mathcal{X}, x}^{-1}(\cdot), \dots, \rho_K \circ \text{Exp}_{\mathcal{F}_K, y_K} \circ f \circ \text{Exp}_{\mathcal{X}, x}^{-1}(\cdot))$, for some $y_k \in \mathcal{Y}_k$ (where $k = 1, \dots, K$), satisfies the estimate:

$$\sup_{x \in K} d_{\prod_k \mathcal{Y}_k}(f(z), \hat{f}'(z)) < \tilde{\omega}_{\rho}^{-1}(2^{-1}\epsilon). \quad (97)$$

We may now derive the estimate (23). We compute:

$$\begin{aligned} \sup_{x \in \mathcal{X}} d_{\mathcal{Y}}(f(x), \rho \circ \hat{f}' \circ \phi(x)) &\leq \sup_{x \in \mathcal{X}} d_{\mathcal{Y}}(H_{t_{\frac{\epsilon}{2}}}(f(x)), f(x)) + d_{\mathcal{Y}}(H_{t_{\frac{\epsilon}{2}}}(f(x)), \rho \circ \hat{f}' \circ \phi(x)) \\ &\leq \sup_{y \in \mathcal{Y}} d_{\mathcal{Y}}(H_{t_{\frac{\epsilon}{2}}}(y), y) + \sup_{x \in \mathcal{X}} d_{\mathcal{Y}}(H_{t_{\frac{\epsilon}{2}}}(f(x)), \rho \circ \hat{f}' \circ \phi(x)) \\ &\leq 2^{-1}\epsilon + \sup_{x \in \mathcal{X}} d_{\mathcal{Y}}(H_{t_{\frac{\epsilon}{2}}}(f(x)), \rho \circ \hat{f}' \circ \phi(x)) \\ &= 2^{-1}\epsilon + \sup_{x \in \mathcal{X}} d_{\mathcal{Y}}(\rho \circ R \circ H_{t_{\frac{\epsilon}{2}}} \circ f \circ \phi^{-1} \circ \phi(x), \rho \circ \hat{f}' \circ \phi(x)) \\ &= 2^{-1}\epsilon + \sup_{z \in \phi(\mathcal{X})} d_{\mathcal{Y}}(\rho \circ R \circ H_{t_{\frac{\epsilon}{2}}} \circ f \circ \phi^{-1}(z), \rho \circ \hat{f}'(z)) \\ &= 2^{-1}\epsilon + \sup_{z \in \phi(\mathcal{X})} \tilde{\omega}_{\rho} \left(d_{\mathcal{Y}}(R \circ H_{t_{\frac{\epsilon}{2}}} \circ f \circ \phi^{-1}(z), \rho \circ \hat{f}'(z)) \right) \\ &= 2^{-1}\epsilon + \sup_{z \in \phi(\mathcal{X})} \tilde{\omega}_{\rho}(\tilde{\omega}_{\rho}^{-1}(2^{-1}\epsilon)) \end{aligned} \quad (98)$$

$$= 2^{-1}\epsilon + 2^{-1}\epsilon = \epsilon; \quad (99)$$

where the inequality transitioning from (98) to (99) follows from (Ebrechts and Hofert, 2013, Proposition 1 (4)) since $t \mapsto \tilde{\omega}_{\rho}$ is continuous and therefore $2^{-1}\epsilon \in (0, \sup_{t \in [0, \infty)} \tilde{\omega}_{\rho}(t)) \cap \{\tilde{\omega}_{\rho}(s) : s \in [0, \infty)\}$. \blacksquare

B.6 Proofs of Applications

B.6.1 APPLICATIONS IN SECTION 3.4

The proofs of Corollaries 23 and 24 are analogous.

Proof [Proof of Corollary 23] Since \mathbb{D}_c^p (resp. \mathbb{D}_c^m) is a Cartan-Hadamard manifold, then Corollary 12 implies that $\mathcal{U}_f(x) = \infty$ for every $f \in C(\mathbb{D}_c^p, \mathbb{D}_c^m)$. The result now follows from Theorem 9 and the representation (9). ■

Proof [Proof of Corollary 24] The result is a direct consequence of Theorem 20 and the representation (9). ■

The proofs of Corollaries 25 and 26 are identical up to their last step.

Proof [Proof of Corollary 25] The main result of Pennec et al. (2006) showed that $d_+(X^\top AX, X^\top BX) = d_+(A, B)$ for every $X \in O_p$ and every $A, B \in P_m^+$. Since both $f(A), \hat{f}(A) \in P_m^+$ for each $A \in P_p^+$ then (14) reduces to

$$\max_{A \in K} \sup_{X \in O_p} d_+(X^\top f(x)X, X^\top \hat{f}(x)X) = \max_{A \in K} d_+(f(A), \hat{f}(A)). \quad (100)$$

The result now follows directly from Theorem 9 applied to the right-hand side of (100); in view of the representation (13). ■

Proof [Proof of Corollary 26] The main result of Pennec et al. (2006) showed that $d_+(X^\top AX, X^\top BX) = d_+(A, B)$ for every $X \in O_p$ and every $A, B \in P_m^+$. Since both $f(A), \hat{f}(A) \in P_m^+$ for each $A \in P_p^+$ then (14) reduces to

$$\max_{A \in K} \sup_{X \in O_p} d_+(X^\top f(x)X, X^\top \hat{f}(x)X) = \max_{A \in K} d_+(f(A), \hat{f}(A)). \quad (101)$$

The result now follows directly from Theorem 20 applied to the right-hand side of (101); in view of the representation (13). ■

Proof [Proof of Corollary 27] The Quarter-Pinched Sphere Theorem of Klingenberg (1968), yields the estimate $\pi \leq \text{inj}_{S^k}(x)$ for any $x \in S^k$ and $k \in \{n, m\}$. The remaining statement then follows directly from Theorem 9. ■

Proof [Proof of Corollary 29] We only need to demonstrate that 1_{S^m} is not null-homotopic. Recall that a space is, by definition, contractible if and only if its identity function is homotopic to a constant function. However, by the first corollary to Hopf's Theorem (Fomenko and Fuchs, 2016, page 125)¹² S^m is not contractible and therefore 1_{S^m} does not lie in the same homotopy class as any constant function. Hence, setting $\mathcal{X} = S^m$, $S^m = \mathcal{Y}$, and $f = 1_{S^m}$ in Theorem 5 yields the result. ■

B.6.2 APPLICATIONS IN SECTION 4.4

Proof [Proof of Corollary 38] Set $\phi = 1_{\mathbb{R}^p}$, $\mathcal{X} = \mathcal{X}_0$, $K = 1$, $\mathcal{F}_1 = \mathcal{Y}_1 = \mathcal{Y}$, and let ρ be the softmax function. Then, together, Example 13 and Theorem 37 yield the result. ■

12. There is no numbering to the results in Fomenko and Fuchs (2016).

Proof [Proof of Corollary 39] Set $\phi = \phi_{\mathcal{G}_n}$, $K = 1$, $\mathcal{X} = \mathcal{G}_n$, $\mathbb{R}^{n(n+1)/2} = \mathcal{X}_0$, $\mathcal{F}_1 = \mathcal{Y}_1 = \mathbb{R}^{m(m+1)/2}$, and $\mathcal{Y} = \mathcal{G}_m$, and $\rho = \phi_{\mathcal{G}_m}^{-1}$. Then, the conclusion is implied by Theorem 37. ■

Proof [Proof of Corollary 40] Set $\phi = 1_{\mathbb{R}^p} \times 1_{\mathbb{R}^a} \times \phi_{\mathcal{G}_n}$, $K = 1$, $\mathcal{X} = \mathcal{G}_n$, $\mathbb{R}^{p+a+n(n+1)/2} = \mathcal{X}_0$, $\mathcal{F}_1 = \mathcal{Y}_1 = \mathbb{R}^{n(n+1)/2}$, and $\mathcal{Y} = \mathcal{G}_n$, and $\rho = \phi_{\mathcal{G}_n}^{-1}$. Then, the conclusion is implied by Theorem 37. ■

Both Corollaries 43 and 44 are a joint consequence of Theorem 37. Therefore, it seems most natural to us to merge their proofs.

Proof [Proof of Corollaries 43 and 44] Set $\phi = 1_{\mathbb{R}^p}$, $K = 1$, $\mathbb{R}^p = \mathcal{X} = \mathcal{X}_0$, $K = 1$, $\mathcal{F}_1 = \mathcal{Y}_1 = \mathcal{Y} = \mathbb{R}^m$, and $\rho = 1_{\mathbb{R}^m}$. Then, the conclusion is implied by Theorem 37. ■

Proof [Proof of Corollary 45] By Proposition 19, if \mathbb{X} is a finite set, then \mathbb{X} is efficient. The conclusion is therefore implied by Corollary 44. ■

Appendix C. Proofs of Topological Obstruction Theorems of Section 3.1

C.1 Proof of Theorem 5

The proofs in this section are independent of the others. They rely on some tools from *algebraic topology*; a subject, which due to its interconnectedness with group theory and other topics in general topology, cannot realistically be comprehensively summarized within the confines of a single paper's appendix. Nevertheless, we refer the interested reader to the following extremely well-written book on the subject: [Hatcher \(2002\)](#).

Lemma 65 (Geometry of $C(\mathcal{X}, \mathcal{Y})$) *Let \mathcal{X} and \mathcal{Y} be complete Riemannian manifolds and let $f_1, f_2 \in C(\mathcal{X}, \mathcal{Y})$. Define the subsets $C_i \subseteq C(\mathcal{X}, \mathcal{Y})$ by: $g \in C_i$ if and only if there exists a (homotopy) $h_g \in C([0, 1] \times \mathcal{X}, \mathcal{Y})$*

$$h_g(0, \cdot) = g \text{ and } h_g(1, \cdot) = f_i.$$

Then, the sets C_1 , and C_2 are closed in $C(\mathcal{X}, \mathcal{Y})$. Moreover, if there does not exist a (homotopy) $h \in C([0, 1] \times \mathcal{X}, \mathcal{Y})$ satisfying:

$$h(0, \cdot) = f_1 \text{ and } h(1, \cdot) = f_2,$$

then, $C_1 \cap C_2 = \emptyset$. Therefore, $C(\mathcal{X}, \mathcal{Y})$ is disconnected.

Proof Note that the sets C_1 and C_2 are precisely the definition of the homotopy class of f_1 and f_2 . Thus, we want to show that the homotopy classes in $C(\mathcal{X}, \mathcal{Y})$ for disjoint closed subsets thereof.

Since \mathcal{Y} is a metric space then ([Munkres, 2000](#), Theorem 46.8) implies that the uniform topology on $C(\mathcal{X}, \mathcal{Y})$ and the compact-open topology thereon, generated by the sub-basis open sets $\{U_{K,O} : K \subseteq \mathcal{X} \text{ compact and } O \subseteq \mathcal{Y} \text{ open}\}$ where:

$$U_{K,O} \triangleq \{f \in C(\mathcal{X}, \mathcal{Y}) : f(K) \subseteq O\},$$

coincide. We therefore consider $C(\mathcal{X}, \mathcal{Y})$ with the latter of the two topologies.

Since \mathcal{X} and \mathcal{Y} are smooth manifolds (without boundary) then by ([Hatcher, 2002](#), Example 0.3) both \mathcal{X} and \mathcal{Y} are CW-complexes (see ([Hatcher, 2002](#), page 5)). Moreover, since \mathcal{X} is a Riemannian manifold then it is a metric space, with (Riemannian) distance function $d_{\mathcal{X}}$; and in-particular so is \mathcal{X} . Furthermore, since \mathcal{X} is closed and bounded in \mathcal{X} and since \mathcal{X} is complete then by the Hopf-Rinow Theorem it is compact. Hence, ([Milnor, 1959](#), Corollary 2) applies and therefore $C(\mathcal{X}, \mathcal{Y})$ is a CW-complex whose path components correspond the homotopy classes in $C(\mathcal{X}, \mathcal{Y})$. By ([Hatcher](#),

2002, Proposition A.4) every CW-complex is locally-contractible and therefore (see (Hatcher, 2002, page 522)) it is locally path-connected. Hence, by (Munkres, 2000, Theorem 25.5) every path component of $C(\mathcal{X}, \mathcal{Y})$ is a connected component and therefore every path component of $C(\mathcal{X}, \mathcal{Y})$ is closed. Hence, every homotopy class in $C(\mathcal{X}, \mathcal{Y})$ is closed. ■

Proof [Proof of Lemma 3] Direct consequence of Lemma 65 and the fact that disjoint closed sets in a metric space (such as $C(\mathcal{X}, \mathcal{Y})$) are separated by a positive distance. ■

Proof [Proof of the claim in Example 3] By Serre’s Finiteness Theorem (see (Ravenel, 1986, Ravenel 86, Chapter I, Lemma 1.1.8)) there exists countably infinite number of distinct homotopy classes in $C(S^3, S^2)$. Thus, there exists an infinite $\mathcal{F} \subset C(S^3, S^2)$ whose elements are all in distinct homotopy classes. Thus, the claim now follows from Lemma 65. ■

Proof [Proof of Theorem 5] Under Assumption 2, by arguing analogously to the proof of Lemma B.1 (mutatis mutandis) we find that for every $g \in C(\mathbb{R}^p, \mathbb{R}^m)$, the map $\rho_\zeta^{-1} \circ g \circ \phi_\alpha$ is well-defined and continuous on $C(U_\alpha, \mathcal{Y})$. In particular, it is continuous on any non-empty compact subset of thereof. Therefore (i) holds. The remainder of the proof is devoted to showing (ii).

Step 1 - Functions Of Different Homotopy Types are in Different Connected Components of $C(\mathcal{X}, \mathcal{Y})$: By Lemma 65, the homotopy classes in $C(\mathcal{X}, \mathcal{Y})$ are disjoint closed subsets.

Step 2 - GDNs are Homotopic to Constant Functions: Every function $\hat{f} \in C(\mathcal{X}, \mathcal{Y})$ of the form $\hat{f} = \rho_\zeta^{-1} \circ g \circ \phi_\alpha$, for some $y \in \mathcal{Y}$, and some $g \in C(\mathbb{R}^p, \mathbb{R}^m)$, is homotopic to a constant function. Observe that for any $t \in [0, 1]$ the function $g_t(x) \triangleq tg(x)$, simply corresponds to re-scaling last affine layer map defining g ; thus, $g_t \in C(\mathbb{R}^p, \mathbb{R}^m)$. Moreover, since multiplication is continuous in \mathbb{R}^m and the composition of continuous functions is again continuous. Thus,

$$G : [0, 1] \times \mathbb{R}^p \rightarrow \mathbb{R}^m \\ (t, x) \mapsto g_t(x),$$

is itself continuous. In particular the restriction of G to $[0, 1] \times \phi_\alpha(U_\alpha)$ is continuous and takes values in \mathcal{Y} . Hence, G defines a homotopy from g to the constant function $g_0(x) \mapsto \rho_\zeta^{-1}(0)$. Therefore, for every $y \in \mathcal{Y}$, and every $g \in C(\mathbb{R}^p, \mathbb{R}^m)$, the function $\hat{f} : x \mapsto \rho_\zeta^{-1} \circ g$ is homotopic to a constant function. Applying by the remark on (Fomenko and Fuchs, 2016, page 26) we conclude that $g \in C(\mathbb{R}^p, \mathbb{R}^m)$, the function $\hat{f} : z \mapsto \rho_\zeta^{-1} \circ g \circ \phi_\alpha(z)$ is homotopic to a constant function.

Step 3 - The \hat{f} and the target function f lie in disjoint closed subsets of $C(\mathcal{X}, \mathcal{Y})$: Now, Step 2 implies that every $\hat{f} \in C(\mathcal{X}, \mathcal{Y})$ of the form (102) is homotopic to the constant function. By hypothesis, f is not homotopic to the constant function. Hence, Step 1 guarantees that \hat{f} and f belong to disjoint connected components of $C(\mathcal{X}, \mathcal{Y})$.

Step 4 - Conclusion: Again applying (Munkres, 2000, Theorem 46.8), the thus must exist some $0 < \epsilon$ bounding the uniform distance by these sets; i.e., (ii) holds. ■

C.2 Proof of Theorem 6

Our goal is to apply Theorem 5 to draw out the conclusion. To this end, it is enough for us to demonstrate the existence of a function in $C(\mathcal{X}, \mathcal{Y})$ which is not null-homotopic.

In the following, given a topological space X , we use $H_k(X)$ (resp. $H^k(X; \mathbb{Z})$, $H^k(X)$) to denote X ’s k^{th} singular homology group (resp. relative homology group with coefficients in \mathbb{Z} , resp. cohomology group). We refer the reader to (Hatcher, 2002, pages 108 and 190) for definitions.

Proof Step 1 - Reduction to Spherical Input Spaces: Since U_α is homeomorphic to \mathbb{R}^p , via ϕ_α ; then, for any $1 \leq d \leq p$, let $S^d(0, 1) \triangleq \{u \in \mathbb{R}^p : \|u\| = 1\}$, and note that $S^d(0, 1)$ is homeomorphic to S^d and it is contained within $\phi_\alpha(U_\alpha) = \mathbb{R}^p$. Let $\mathcal{X}_d \triangleq \phi_\alpha^{-1}(S^d(0, 1)) \subset U_\alpha$, and note that \mathcal{X}_d is must also be homeomorphic to S^d . Note that, for every $1 \leq d \leq p$, since $\mathcal{X}_d \subset U_\alpha$ then for any $\zeta \in Z$ and any continuous function $g : \mathbb{R}^p \rightarrow \mathbb{R}^m$ the map $\rho_\zeta^{-1} \circ g \circ \phi_\alpha$ is well-defined. In particular, since the activation function σ is continuous then, for every $k \in \mathbb{N}$, every $g \in C(\mathbb{R}^p, \mathbb{R}^m)$ is a continuous function from \mathbb{R}^p to \mathbb{R}^m . Hence, for every $k \in \mathbb{N}_+$, every $g \in C(\mathbb{R}^p, \mathbb{R}^m)$, and every $y \in \mathcal{Y}$, the map

$$\hat{f} \triangleq \rho_\zeta^{-1} \circ g \circ \phi_\alpha, \quad (102)$$

is well-defined on each \mathcal{X}_d for $1 \leq d \leq p$.

Step 2 - Existence of a Non-Null Homotopic Function in $C(\mathcal{X}, \mathcal{Y})$: By the remark on (Fomenko and Fuchs, 2016, page 26), for any $1 \leq d \leq p$, there is a continuous function in $C(\mathcal{X}_d, \mathcal{Y})$ which is not homotopic to a constant function if and only if there is a continuous function in $C(S^d, \mathcal{Y})$ which is not homotopic to a constant function. Our next step is to demonstrate that there is a $1 \leq d \leq p$ for which $C(S^d, \mathcal{Y})$ contains a continuous function which is not homotopic to a constant function. Once we prove that d exists, we set $\mathcal{X} \triangleq \mathcal{X}_d$.

Since, \mathcal{Y} is closed and orientable then (Hatcher, 2002, Theorem 3.30 (c)) implies that the (singular) homology groups $H_s(\mathcal{Y})$ are all trivial for $s > p$. Suppose that no such d existed. Then, $C(S^d, \mathcal{Y})$ only consists of null-homotopic maps (maps which are homotopic to a constant function) and (Spanier, 1995b, Proposition 7.2) implies that for every $d < p$ the (singular) homology groups $H_d(\mathcal{Y}; \mathbb{Z})$ are all trivial $d < p$. Furthermore, (Hatcher, 2002, Theorem 3.26) implies that $H_d(\mathcal{Y})$ is isomorphic to \mathbb{Z} and therefore it is non-trivial. We may therefore apply (Spanier, 1995b, Hurewicz's Isomorphism Theorem; Proposition 7.2) to find that $\pi(S^d, \mathcal{Y})$ is non-trivial since $H_p(\mathcal{Y})$ is non-trivial. By the same result, there exists a surjective (group homomorphism) from $\pi(S^p, \mathcal{Y})$ onto $H_p(\mathcal{Y})$ and therefore $\pi(S^p, \mathcal{Y})$ must be non-trivial; a contradiction. Hence, such a $d \in \{1, \dots, p\}$ must exist. Hence, $C(\mathcal{X}, \mathcal{Y}) = C(\mathcal{X}_d, \mathcal{Y})$ contains a function which is not null-homotopic. **Step 3 - Conclusion:** The result now follows by Theorem 5 (ii). ■

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